Quantification of Entanglement with Witness Operators

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A multipartite state $\rho$ shared by $N$ parties is

- separable if

$$\rho_{1...N} = \sum_i p_i |\psi_i\rangle_1 \langle \psi_i | \otimes ... \otimes |\psi_i\rangle_N \langle \psi_i |,$$

- $k$-producible if

$$\rho_{1...N} = \sum_i p_i |\phi_i\rangle_1 \langle \phi_i | \otimes ... \otimes |\phi_i\rangle_k \langle \phi_i |,$$

where the $|\phi_i\rangle_j$ are states of maximally $k$ original parties

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A quantum state is

- entangled if it is not separable
- $k$-entangled if it is not $k$-producible

Given a particular density operator $\rho$, how to know if it is entangled, or $k$-entangled?
For every entangled state $\rho$ there is a Hermitian operator $W$ which \textit{witness} its entanglement

- $\text{tr}(W\rho) < 0$, $\text{tr}(W\sigma) \geq 0$ for all separable $\sigma$

Can entanglement witnesses also be used to quantify entanglement?

We call *witnessed entanglement* any quantifier which can be expressed as

$$E(\rho) = \max \left\{ 0, - \min_{W \in \mathcal{M}} \text{Tr}(W \rho) \right\}, \quad \mathcal{M} = \mathcal{W} \cap \mathcal{C}$$

- $\mathcal{W}$: set of entanglement witnesses
- $\mathcal{C}$: set such that the minimization problem is well-defined

Several entanglement quantifiers are included in this picture

- **Negativity**
  \[
  N(\rho) = \max \{ 0, - \min_{0 \leq W \leq I} Tr(W^{T_A} \rho) \}
  \]

- **Fidelity of distillation under PPT operations**
  \[
  F_d(\rho) = \frac{I}{d} + \max \{ 0, - \min_{W \in M} Tr(W^{T_A} \rho) \}
  \]
  \[
  M = \{ W \mid (1 - d)I/d \leq W \leq I/d, \ 0 \leq W^{T_A} \leq 2I/d \}
  \]

- **Concurrence**
  \[
  C(\rho) = \max \{ 0, - \min_{A \in SL(2, C)} Tr((|A\rangle\langle A|)^{T_B} \rho) \}
  \]
  \[
  |A\rangle = (A \otimes I)(\sum_i |ii\rangle), \quad \det(A) = 1
  \]
Multipartite Entanglement

Witnessed Entanglement

Numerical Calculation

Applications

Generalized Robustness

Multipartite Entanglement Hierarchy

\[
E(\rho) = \max \left\{ 0, -\min_{W \in \mathcal{M}} Tr(W \rho) \right\}, \quad \mathcal{M} = \mathcal{W} \cap \mathcal{C}
\]

If the set \( \mathcal{C} \) is convex, \( E(\rho) \) can be written in a dual formulation.

- If \( \mathcal{M} = \{ W \in \mathcal{W} \mid W \leq I \} \), \( E(\rho) \) is equal to the generalized robustness of entanglement \( R_G \), i.e., the minimum \( s \) such that

  \[
  \rho + s\pi
  \]

  is separable, where \( \pi \) is an arbitrary state.

- If \( \mathcal{M} = \{ W \in \mathcal{W} \mid W \geq -I \} \), \( E(\rho) \) is equal to the best separable approximation measure \( BSA = 1 - \lambda \), where \( \lambda \) is the maximum number such that

  \[
  \rho = \lambda \sigma + (1 - \lambda)\delta \rho,
  \]

  where \( \sigma \) is separable and \( \delta \rho \) an arbitrary state.
New entanglement quantifiers can be formed by changing the set $\mathcal{C}$. For example:

If $\mathcal{M} = \{ W \in \mathcal{W} | -nl \leq W \leq ml \}$, we get an infinity family of new entanglement monotones $E_{n:m}$.

The dual representation of $E_{m:n}(\rho)$ is

\[
\begin{align*}
\text{minimize} & \quad ms + nt \\
\text{subject to} & \quad \rho + s\pi_1 = (1 + s - t)\sigma + t\pi_2
\end{align*}
\]

where $\pi_i$ are density matrices, $\sigma$ is a separable state and $s, t \geq 0$. From (1) we find that

$$\lim_{m \to \infty} E_{n:m}(\rho) = nBSA(\rho), \quad \lim_{n \to \infty} E_{n:m}(\rho) = mR_G(\rho)$$
\( k \)-partite entanglement can be quantified as follows

\[
E^k(\rho) = \max \left\{ 0, - \min_{W \in \mathcal{M}} \text{Tr}(W \rho) \right\}, \quad \mathcal{M} = \mathcal{W}_k \cap \mathcal{C}
\]

where now \( \mathcal{W}_k \) is the set of \( k \)-entanglement witness (the set of operators which are positive on all \( k \)-producible states)

All the constraints imposed to a \( m \)-EW are imposed to a \( n \)-EW with \( n \geq m \).

\[
E^m(\rho) \geq E^n(\rho), \quad \forall \quad n \geq m.
\]
The calculation of a given measure of entanglement for a general state is usually extremely hard.

The computation complexity of the separability problem for bipartite mixed states was even proved to be NP-HARD!

Recently, methods from convex relaxation theory have been applied to the separability problems.

All those methods can be used to calculate, in an approximative manner, every multipartite *witnessed entanglement*.

Example:

\[ \rho_q = q|W\rangle\langle W| + (1 - q)|GHZ\rangle\langle GHZ|, \quad 0 \leq q \leq 1 \]

**Figure:** $E_{n:1}^2(\rho_q)$ for $0 \leq n \leq 4$ and $0 \leq q \leq 1$. 
Are all maximally entangled states pure?

- The answer depends on which entanglement measure we are dealing with
- Yes for the $R_G$, no for the BSA.
Discontinuities in the entanglement of some family of states induced by the geometry of the separable states.

These abstract phase transitions can be 'experimentally' tested using entanglement witnesses.

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Entanglist Corrections to the Currie-Law

The magnetic susceptibility at very low temperatures can be approximated by

$$\chi \approx (g^2 \mu_B^2 / kT)[NI + (1/3) \sum_i \vec{\sigma}_i \cdot \vec{\sigma}_{i+1}]$$

The optimal entanglement witness for the ground state if the XXX Heisenberg model is

$$W = \left( NI + \sum_{i=1}^{N} \left( \sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \sigma_i^z \sigma_{i+1}^z \right) \right) / 2N,$$

Then,

$$\chi \approx \frac{2Ng^2 \mu_B^2}{3kT} + \frac{2NR_G}{3kT}.$$
Miscellaneous applications

- Upper bounds for the *distillable entanglement*
- Lower bounds for the *entanglement of formation*
- Systems subjected to superselection rules
- Gaussian states

Fidelity of Teleportation

- How good is a bipartite state as a quantum channel?

\[ f(\rho) = \sup_{\Gamma \in \text{LOCC}} \int d\psi \langle \psi | \Gamma(\rho \langle \psi \rangle \langle \psi |) \langle \psi \rangle, \]

- This fidelity of teleportation is related to the singlet fraction:

\[ f = \frac{dF_d + 1}{d + 1}, \quad F_d(\rho) = \sup_{\Omega \in \text{LOCC}} \text{tr}[\Omega(\rho)\phi_d], \]

How helpful can an entangled state be in increasing the fidelity of teleportation of another state?

\[ E_d(\sigma) = \max_{\rho \in \mathcal{D}(\mathcal{H})} \frac{F_d(\rho \otimes \sigma) - F_d(\rho)}{F_d(\rho)}, \quad E(\sigma) = \sup_{d \in \mathbb{N}, d \geq 2} E_d(\sigma) \]

We can find an operational interpretation for the robustness. If \( \sigma \) acts on \( \mathbb{C}^m \otimes \mathbb{C}^n \), \( m \leq n \), then

\[ R_G(\sigma) = E_m(\sigma) = E(\sigma) \]

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Thank you!

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