Quantification of Entanglement with Witness Operators

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Multipartite Entanglement

A multipartite state ρ shared by N parties is

- separable if

$$\rho_{1\ldots N} = \sum_i p_i |\psi_i\rangle_1\langle\psi_i| \otimes \ldots \otimes |\psi_i\rangle_N\langle\psi_i|,$$

- k-producible if

$$\rho_{1\ldots N} = \sum_i p_i |\phi_i\rangle_1\langle\phi_i| \otimes \ldots \otimes |\phi_i\rangle_K\langle\phi_i|,$$

where the $|\phi_i\rangle_j$ are states of maximally k original parties

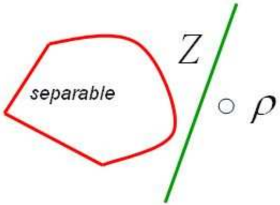
O. Guehne, G. Toth, H.J. Briegel, quant-ph/0502160

A quantum state is

- entangled if it is not separable
- k-entangled if it is not k-producible

Given a particular density operator ρ, how to know if it is entangled, or k-entangled?

- For every entangled state ρ there is a Hermitian operator W which *witness* its entanglement
- $tr(W\rho) < 0, \qquad tr(W\sigma) \geq 0$ for all separable σ



- Can entanglement witnesses also be used to quantify entanglement?

M. Horodecki, P. Horodecki, R. Horodecki, Phys. Lett. A **223**, 1 (1996)

Witnessed Entanglement

- We call *witnessed entanglement* any quantifier which can be expressed as

$$E(\rho) = \max\{0, -\min_{W \in \mathcal{M}} Tr(W\rho)\}, \qquad \mathcal{M} = \mathcal{W} \cap \mathcal{C}$$

- $\mathcal{W}$: set of entanglement witnesses
- $\mathcal{C}$: set such that the minimization problem is well-defined

F.G.S.L. Brandão, Phys. Rev. A **72**, 022310 (2005)

Several entanglement quantifiers are included in this picture

- Negativity

$$\mathcal{N}(\rho) = \max\{0, -\min_{0 \leq W \leq I} Tr(W^{T_A}\rho)\}$$

- Fidelity of distillation under PPT operations

$$F_d(\rho) = \frac{I}{d} + \max\{0, -\min_{W \in \mathcal{M}} Tr(W^{T_A}\rho)\}$$

$$\mathcal{M} = \{W \mid (1-d)I/d \leq W \leq I/d, \quad 0 \leq W^{T_A} \leq 2I/d\}$$

- Concurrence

$$C(\rho) = \max\{0, -\min_{A \in SL(2,C)} Tr((|A\rangle\langle A|)^{T_B}\rho)\}$$

$$|A\rangle = (A \otimes I)(\sum_i |ii\rangle), \qquad \det(A) = 1$$

$$E(\rho) = \max\{0, -\min_{W\in\mathcal{M}} Tr(W\rho)\}, \quad \mathcal{M} = \mathcal{W}\cap\mathcal{C}$$

If the set $\mathcal{C}$ in convex, $E(\rho)$ can be written in a dual formulation.

- If $\mathcal{M} = \{W \in \mathcal{W} \mid W \leq I\}$, $E(\rho)$ is equal to the generalized robustness of entanglement R_G, i.e the minimum s such that

$$\rho + s\pi$$

  is separable, where π is an arbitrary state.
- If $\mathcal{M} = \{W \in \mathcal{W} \mid W \geq -I\}$, $E(\rho)$ is equal to the best separable aproximation measure $BSA = 1-\lambda$, where λ is the maximum number such that

$$\rho = \lambda\sigma + (1-\lambda)\delta\rho,$$

  where σ is separable and $\delta\rho$ an arbitrary state.

- New entanglement quantifiers can be formed by changing the set $\mathcal{C}$. For example:
- If $\mathcal{M} = \{W \in \mathcal{W} \mid -nI \leq W \leq mI\}$, we get an infinity family of new entanglement monotones $E_{n:m}$
- The dual representation of $E_{m:n}(\rho)$ is

$$\begin{aligned} \text{minimize} \quad & ms + nt \\ \text{subject to} \quad & \rho + s\pi_1 = (1 + s - t)\sigma + t\pi_2 \end{aligned} \tag{1}$$

where π_i are density matrices, σ is a separable state and $s, t \geq 0$. From (1) we find that

$$\lim_{m\to\infty} E_{n:m}(\rho) = nBSA(\rho), \quad \lim_{n\to\infty} E_{n:m}(\rho) = mR_G(\rho)$$

- k-partite entanglement can be quantified as follows

$$E^k(\rho) = \max\{0, -\min_{W \in \mathcal{M}} Tr(W\rho)\}, \qquad \mathcal{M} = \mathcal{W}_k \cap \mathcal{C}$$

where now $\mathcal{W}_k$ is the set of k-entanglement witness (the set of operators which are positive on all k-producible states)

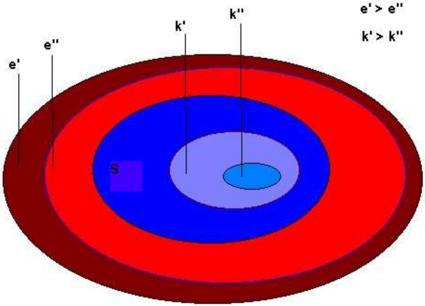
- All the constraints imposed to a m-EW are imposed to a n-EW with $n \geq m$.

$$E^m(\rho) \geq E^n(\rho), \quad \forall \;\; n \geq m.$$

Numerical Calculation

- The calculation of a given measure of entanglement for a general state is usually extremely hard
- The computation complexity of the separability problem for bipartite mixed states was even proved to be NP-HARD!
- Recently, methods from convex relaxation theory have been applied to the separability problems
- All those methods can be used to calculate, in an approximative manner, every multipartite *witnessed entanglement*

L. Gurvits, in *Proccedings of the 35th ACM Symposium on the Theory of Computing*, pp. 10.



A.C. Doherty, P.A. Parrilo, F.M. Spedalieri, Phys. Rev. Lett. **88**, 187904 (2002)
F.G.S.L. Brandão, R.O. Vianna, Phys. Rev. Lett. **93**, 220503 (2004)
J. Eisert, P. Hyllus, O. Guehne, M. Curty, Phys. Rev. A **70**, 062317 (2004)

- Example:

$$\rho_q = q|W\rangle\langle W| + (1-q)|GHZ\rangle\langle GHZ|, \quad 0 \leq q \leq 1$$

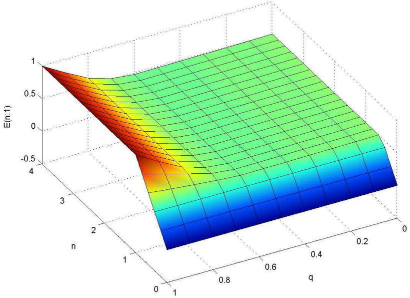
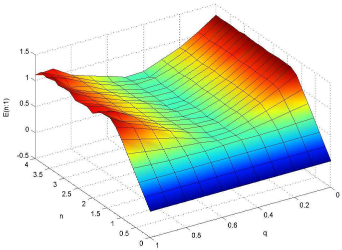


Figure: $E^2_{n:1}(\rho_q)$ for $0 \leq n \leq 4$ and $0 \leq q \leq 1$.

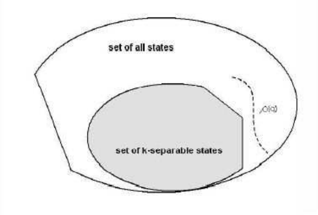
Are all maximally entangled states pure?

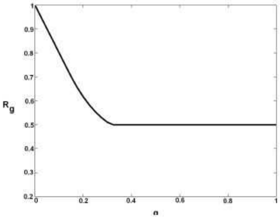
- The answer depends on which entanglement measure we are dealing with
- Yes for the R_G, no for the BSA.



D. Cavalcanti, F.G.S.L. Brandão, M.O. Terra Cunha, Phys. Rev. A 72, 040303(R) (2005)

- Discontinuities in the entanglement of some family of states induced by the geometry of the separable states





- These abstract phase transitions can be 'experimentally' tested using entanglement witnesses

D. Cavalcanti, F.G.S.L. Brandão, M.O. Terra Cunha, quant-ph/0510068

Entanglist Corrections to the Currie-Law

- The magnetic susceptibility at very low temperatures can be approximated by

$$\chi \approx (g^2\mu_B^2/kT)[NI + (1/3)\sum_i \vec{\sigma_i}.\vec{\sigma_{i+1}}]$$

- The optimal entanglement witness for the ground state if the XXX Heisenberg model is

$$W = \left(NI + \sum_{i=1}^{N} (\sigma_i^x\sigma_{i+1}^x + \sigma_i^y\sigma_{i+1}^y + \sigma_i^z\sigma_{i+1}^z) \right) /2N,$$

- Then,

$$\chi \approx \frac{2Ng^2\mu_B^2}{3kT} + \frac{2NR_G}{3kT}.$$

F.G.S.L. Brandão, Phys. Rev. A 72, 022310 (2005)

Miscellaneous applications

- Upper bounds for the *distillable entanglement*
- Lower bounds for the *entanglement of formation*
- Systems subjected to superselection rules
- Gaussian states

F.G.S.L. Brandão, Phys. Rev. A 72, 022310 (2005)

Fidelity of Teleportation

- How good is a bipartite state as a quantum channel?

$$f(\rho) = \sup_{\Gamma \in LOCC} \int d\psi \, \langle\psi|\Gamma_\rho\left(|\psi\rangle\langle\psi|\right)|\psi\rangle,$$

- This fidelity of teleportation is related to the singlet fraction:

$$f = \frac{dF_d + 1}{d+1}, \qquad F_d(\rho) = \sup_{\Omega \in LOCC} tr[\Omega(\rho)\phi_d],$$

P. Horodecki, M. Horodecki, R. Horodecki, Phys. Rev. A **60**, 1888 (1999)

Generalized Robustness

- How helpful can a entangled state be in increasing the fidelity of teleportation of another state?

$$E_d(\sigma) = \max_{\rho \in \mathcal{D}(\mathcal{H})} \frac{F_d(\rho \otimes \sigma) - F_d(\rho)}{F_d(\rho)}, \quad E(\sigma) = \sup_{d \in \mathbb{N}, d \geq 2} E_d(\sigma)$$

- We can find an operational interpretation for the robustness. If σ acts on $\mathbb{C}^m \otimes \mathbb{C}^n$, $m \leq n$, then

$$R_G(\sigma) = E_m(\sigma) = E(\sigma)$$

F.G.S.L. Brandão, quant-ph/0510078

Thank you!

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