New Variants of the Quantum de Finetti Theorem with Applications

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Based on joint work with Aram Harrow

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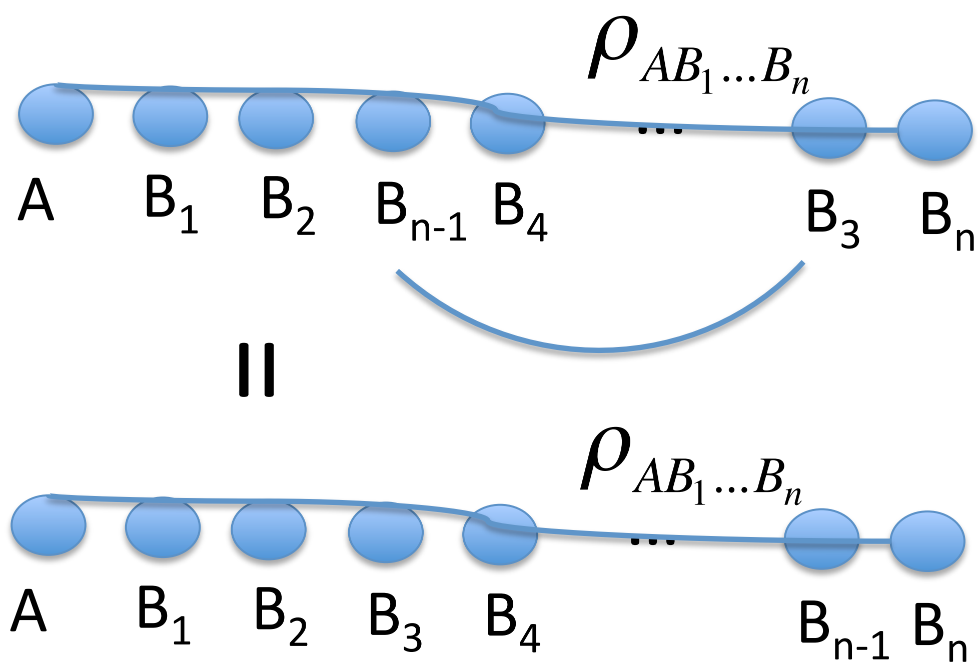
Symmetric States

$\rho_{AB_1\ldots B_n} \in D(A \otimes B_1 \otimes \ldots \otimes B_n)$ is permutation symmetric in the B subsystems if $\rho_{AB_1\ldots B_n} = \rho_{AB_{\pi(1)}\ldots B_{\pi(n)}}$ for every permutation π.



Quantum de Finetti Theorem

(Christandl, Koenig, Mitchson, Renner '05) Let $\rho_{AB_1....B_n}$ be a state symmetric in the B subsystems. Then

$$\min_{\mu} \left\| \rho_{AB_1...B_k} - \int \mu(d\sigma)\rho_\sigma \otimes \sigma^{\otimes k} \right\|_1 \leq \frac{d^2 k}{n}$$

Final installment in a long sequence of works:
(Hudson, Moody '76), (Stormer '69), (Raggio, Werner '89), (Caves, Fuchs, Schack '01), (Koenig, Renner '05), …

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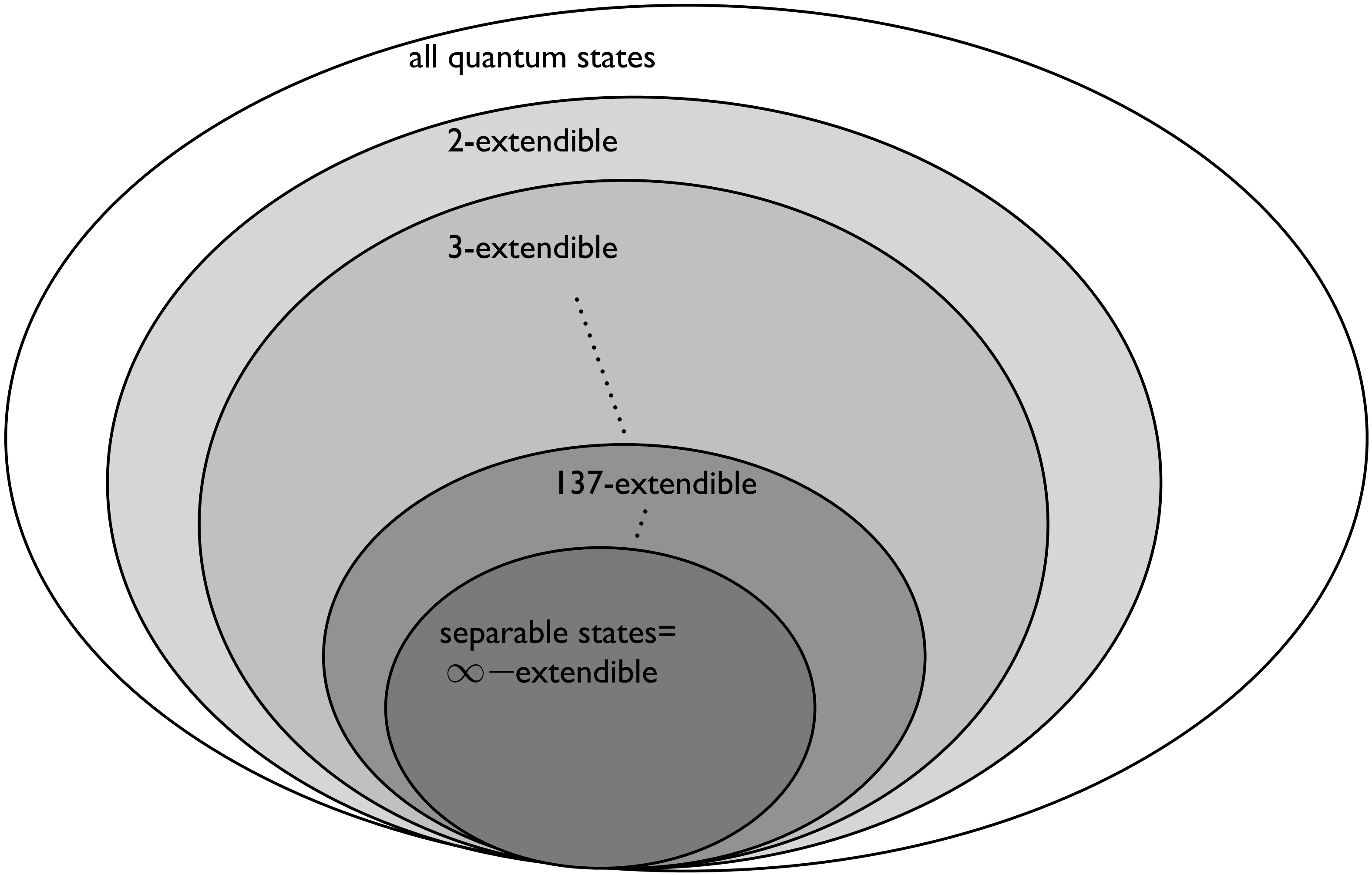
Def. We say ρ_{AB} is *k*-extendible if there is a symmetric extension on B subsystems $\rho_{AB_1....B_n}$ of it.

Quantum de Finetti Theorem as Monogamy of Entanglement



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Way Forward: Prove new versions of de Finetti with better error term, but for a coarser notion of approximation

Relaxed and Improved Quantum de Finetti Theorems

So far we know two examples of this approach:

(Renner '07) Exponential de Finetti Theorem: error term $\exp(-\Omega(n-k))$. Target state convex combination of "almost i.i.d." states.

(B., Christandl, Yard '10): de Finetti theorem for $k = 1$ with error term $O(\log(\dim))$. Error measured in 1-LOCC norm.

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Can we push this approach further? Is it worth doing so?

Outline

- Quantum de Finetti Theorem for *Local* Measurements

  Optimality $O(n^{1/2})$ unentangled proofs for SAT

  Subexponential Algorithm for Small Set Expansion

  Efficient "Pretty Good" Tomography

  Proof

- Quantum de Finetti Theorem *without* Symmetry

  Calculating groundenergy dense local hamiltonians

  ~~Evidence Against Quantum PCP~~

Quantum de Finetti for Local Measurements

Thm Part 1 For any state $\rho_{AB_1 \ldots B_n} \in D(A \otimes B_1 \otimes \ldots \otimes B_n)$ symmetric in the B subsystems and ν a distribution over quantum operations $\{\Lambda_{A,\nu}\}_\nu$:

$$\min_{\sigma \in SEP} \max_{\Lambda_B} \mathop{E}_{\nu} \left\| \Lambda_{A,\nu} \otimes \Lambda_B \left(\rho_{AB} - \sigma_{AB} \right) \right\|_1 \leq \sqrt{\frac{2\ln(2)\log|K|}{n}}$$

with $\Lambda_{A,\nu} : D(A) \rightarrow D(K)$ channels with output dim |K|

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Obs: For $\{\Lambda_{A,\nu}\} = \{id\}$, we recover result of (B.,Christandl, Yard '10) as

$$\max_{\Lambda_B} \left\| I \otimes \Lambda_B \left(\rho_{AB} - \sigma_{AB} \right) \right\|_1 = \left\| \left(\rho_{AB} - \sigma_{AB} \right) \right\|_{1-LOCC}$$

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Obs2: “Semi-classical” quantum de Finetti Thm

Quantum de Finetti for Local Measurements (part 2)

Thm Part 2 For any symmetric state $\rho_{B_1\ldots B_n} \in D(B_1 \otimes \ldots \otimes B_n)$ there is a measure μ s.t.

$$\max_{\Lambda_2,\ldots,\Lambda_k} \left\| I_{B_1} \otimes \Lambda_{B_2} \otimes \ldots \otimes \Lambda_{B_k} \left(\rho_{B_1\ldots B_k} - \int \mu(d\sigma)\sigma^{\otimes k} \right) \right\|_1$$

$$\leq \sqrt{\frac{4k^2 \log|B|}{n-l}}$$

with $\Lambda_{B_j}(X) = \sum_{i=1}^{L} tr(M_{i,B_j} X)|i\rangle\langle i|$ quantum-classical channels

Compare with dimension independent classical de Finetti

(Diaconis, Freedman '80)

Short Quantum Proofs

Given satisfiable 3-SAT instance on n variables, what's the size of the smallest proof for it?

(Remainder 3-SAT: (x_i or x_j or x_k) and … and (x_p or x_q or x_s))

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Classically we need $\Omega(n)$ bits, unless there is a subexponential time algorithm for SAT

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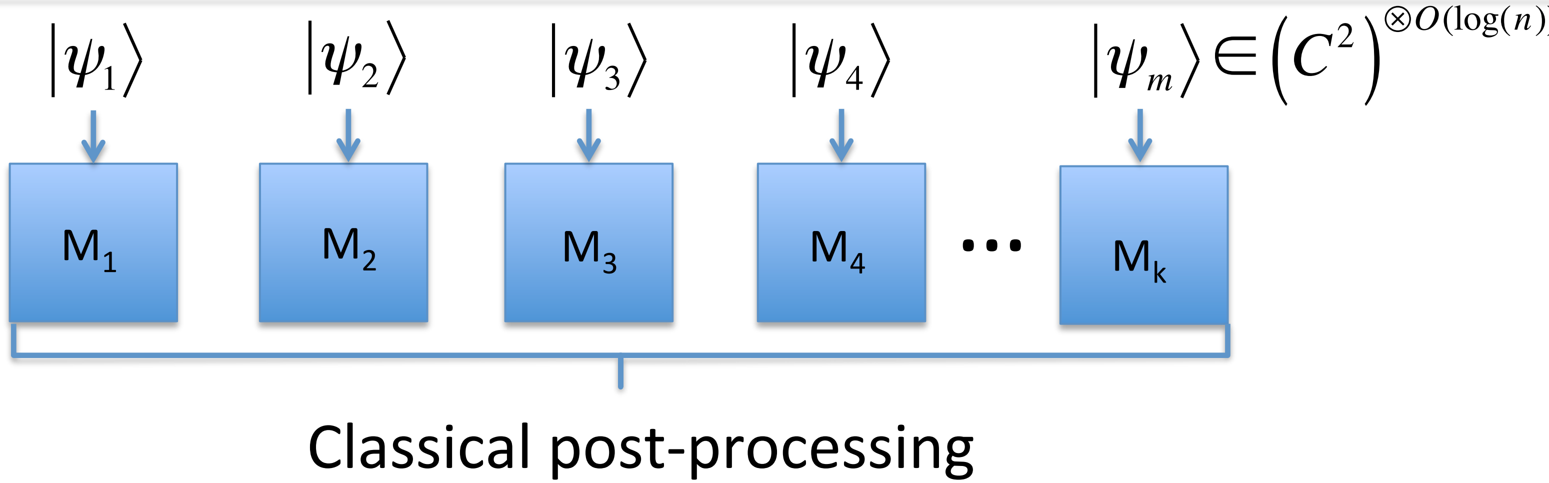
Quantumly we need $\Omega(n)$ qubits, unless there is a quantum subexponential algorithm for SAT (Marriott and Watrous '05)

But what if we have a quantum state, but with the *promise* that parts of it are not entangled?

The Power of Unentanglement

(Chen, Drucker '10, based on Aaronson *et al* '07): One can convince a quantum verifier that a n-variable SAT instance is satisfiable by sending $m = O(n^{1/2}\text{polylog}(n))$ states, each of $O(\log(n))$ qubits, assuming the promise that the states are not entangled with each other.

The verifier only measures locally each state and post-process the classical outcomes.



The Power of Unentanglement (in complexity theory jargon….)

BellQMA$_k$(m, c, s): Analogue of QMA (or NP) where prover sends m unentangled proofs, each of k qubits, to the verifier, who measures each of the proofs and classical post-process the outcomes to decide whether to accept.

- YES instance: there is a proof that makes him accept with prob. > c.
- NO instance: no proof is accepted with prob. > s

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(Chen, Drucker '10) n-variable SAT is in BellQMA$_{O(\log(n))}$($n^{1/2}$polylog(n), 2/3, 1/3)

Implies BellQMA$_k$(m) is not in QMA$_{o(km^{2-\varepsilon})}$ unless there is a subexponential algorithm For SAT. Square root advantage of unentanglement

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Cor. $\text{BellQMA}_k(m)$ is in $\text{QMA}_{O(km^2)}$

The square root advantage is all there is!

Proof Idea: Instead of sending $|\psi\rangle^{\otimes m}$, prover sends $\rho_{A_1 \ldots A_{O(km^2)}}$.
Verifier symmetrizes ρ, traces out all except *m* subsystems and runs original verification protocol. By Thm part 2 this works.

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Thm Part 2 For any symmetric state $\rho_{B_1...B_n} \in D(B_1 \otimes ... \otimes B_n)$ there is a measure μ s.t.

$$\max_{\Lambda_2,...,\Lambda_k} \left\| I_{B_1} \otimes \Lambda_{B_2} \otimes ... \otimes \Lambda_{B_k} \left(\rho_{B_1...B_k} - \int \mu(d\sigma)\sigma^{\otimes k} \right) \right\|_1$$

$$\leq \sqrt{\frac{4k^2 \log|B|}{n-l}}$$

with $\Lambda_{B_j}(X) = \sum_{i=1}^{L} tr(M_{i,B_j} X)|i\rangle\langle i|$ quantum-classical channels

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Optimization over Separable States

For a m-partite matrix M define

$$h_{SEP(m)}(M) := \max_{|\psi_1\rangle,\dots,|\psi_m\rangle} \langle \psi_1,\dots,\psi_m | M | \psi_1,\dots,\psi_m \rangle$$

Acceptance probability of BellQMA(m) is equivalent to estimating $h_{SEP}(M)$ for a "Bell operator" M.

- Chen and Drucker protocol translates into hardness result for estimating $h_{SEP}(M)$

- De Finetti bound translates into algorithms results for $h_{SEP}(M)$

Subexponential Algorithm for Small Set Expansion (in passing)

The result has implications to polynomial optimization:
O(log(n)) rounds of Lasserre hierarchy are enough for optimizing polynomials over the hypersphere. Generalizes similar result by (Powers, Reznick ‘00) for the hypercube.

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(Barak, B., Harrow, Kelner, Steurer, Zhou '12) Algorithm (B, Christandl, Yard '10) and hardness result (Aaronson et al '07 + Harrow, Montanaro '10) for the quantum problem have implications to Small Set Expansion and Unique Games problems: Route to prove quasi-polynomial hardness of unique games and for giving a quasi-polynomial time alg. for it

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Tomography

Suppose we have $\int \mu(d\sigma)\sigma^{\otimes k}$ for unknown μ. We can perform tomography by measuring l copies. Conditioned on obtained outcomes X we get w.h.p. a post-selected state

$$\int \mu_X(d\sigma)\sigma^{\otimes k-l}$$

where up to error $\exp(-l\varepsilon^2)$, μ_X only has support on states that are poly(d)ε-close to a state compatible with statistics.

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Standard de Finetti allows us to apply same reasoning to general ω_n (by symmetrizing it, tracing out n-k copies and measuring l of the remaining k copies). Same conclusion as before, but now μ_X has support on good states up to error $\exp(-l\varepsilon^2) + d^2k/n$.

Tomography

But we need to measure $l = O(\text{poly}(d))$ copies. Exponential in the number of qubits!

Makes sense, since we need an exponential number of parameters (in $\log(d)$) to describe the state.

Can we improve on this?

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Pretty Good Tomography

(Aaronson '06 Pretty-good-tomography thm)

Given $\int \mu(d\sigma)\sigma^{\otimes k}$ and a distribution over measurements ν, suppose we measure l = O(poly(1/ε)log(d)) copies and get outcomes X. Let ρ_X be any state compatible with X. Then w.h.p. the post-selected state can be written as

$$\int \mu_X(d\sigma)\sigma^{\otimes k-l}$$

s.t., up to error ε, μ_X only has support on states σ s.t.

$$\Pr_{M\sim\nu}\left(\left\|M(\sigma)-M(\rho_X)\right\|_1 > \varepsilon^{\alpha}\right) < \varepsilon^{\beta}$$

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Proof (of part 1)

Let $\pi_{AB_1\ldots B_nK} := \mathop{E}_{\mu}\left(\Lambda_{A,\mu}\otimes\Lambda_{B_1}\otimes\ldots\otimes\Lambda_{B_n}\right)\left(\rho_{AB_1\ldots B_n}\right)\otimes|\mu\rangle\langle\mu|_K$

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On one hand:

$$\max_{\Lambda_1,\ldots,\Lambda_n} I(A:B_1\ldots B_n\,|\,K)_\pi = \max_{\Lambda_1,\ldots,\Lambda_n}\mathop{E}_{\mu} I(A:B_1\ldots B_n)_{\pi_\mu} \le \log|K|$$

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On the other hand we will show:

$$\max_{\Lambda_1,\ldots,\Lambda_n} I(A:B_1\ldots B_n\,|\,K)_\pi \geq \frac{n}{2\ln 2}\min_{\sigma\in SEP}\max_{\Lambda}\underset{\mu}{E}\left\|\Lambda_{A,\mu}\otimes\Lambda\left(\rho-\sigma\right)\right\|_1^2$$

Proof (of part 1)

Remember

$$\pi_{AB_1\ldots B_nK} := \mathop{E}_{\mu}\left(\Lambda_{A,\mu}\otimes\Lambda_{B_1}\otimes\ldots\otimes\Lambda_{B_n}\right)\left(\rho_{AB_1\ldots B_n}\right)\otimes|\mu\rangle\langle\mu|_K$$

We have:

$$\max_{\Lambda_1,\ldots,\Lambda_n} I(A:B_1\ldots B_n|K)_\pi$$

$$= \max_{\Lambda_1,\ldots,\Lambda_n}\left(I(A:B_1|K)_\pi+\ldots+I(A:B_n|KB_1\ldots B_{n-1})_\pi\right)$$

$$= \max_{\Lambda_1,\ldots,\Lambda_{n-1}}\left(I(A:B_1|K)_\pi+\ldots+\max_{\Lambda_n} I(A:B_n|KB_1\ldots B_{n-1})_\pi\right)$$

Proof (of part 1)

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$$\pi_{AB_1\ldots B_nK} := \mathop{E}_{\mu}\left(\Lambda_{A,\mu}\otimes\Lambda_{B_1}\otimes\ldots\otimes\Lambda_{B_n}\right)\left(\rho_{AB_1\ldots B_n}\right)\otimes|\mu\rangle\langle\mu|_K$$

We have:

$$\begin{aligned}
&\max_{\Lambda_1,\ldots,\Lambda_n} I(A:B_1\ldots B_n|K)_\pi \\
&= \max_{\Lambda_1,\ldots,\Lambda_n}\left(I(A:B_1|K)_\pi+\ldots+I(A:B_n|KB_1\ldots B_{n-1})_\pi\right) \\
&= \max_{\Lambda_1,\ldots,\Lambda_{n-1}}\left(I(A:B_1|K)_\pi+\ldots+\max_{\Lambda_n} I(A:B_n|KB_1\ldots B_{n-1})_\pi\right)
\end{aligned}$$

Thus it suffices to prove

$$\max_{\Lambda_n} I(A:B_n|KB_1\ldots B_{n-1})_\pi \geq \frac{1}{2\ln 2}\min_{\sigma\in SEP}\max_{\Lambda_n}\mathop{E}_{\mu}\left\|\Lambda_{A,\mu}\otimes\Lambda_{B_n}\left(\rho_{AB_n}-\sigma_{AB_n}\right)\right\|_1^2$$

Proof (of part 1)

Note that $I(A:B_n|KB_1...B_{n-1})_\pi = \underset{\mu}{E}\, I(A:B_k|B_1...B_{k-1})_{\pi_\mu}$

with $\pi_\mu := \left(\Lambda_{A,\mu} \otimes \Lambda_{B_1} \otimes ... \otimes \Lambda_{B_n}\right)\left(\rho_{AB_1...B_n}\right)$

Moreover $I(A:B_n|B_1...B_{n-1})_{\pi_\mu} = \sum_i q_i I(A:B_n)_{\pi_{i,\mu}}$

for $\pi_{i,\mu} := \left(\Lambda_{A,\mu} \otimes \Lambda_{B_k}\right)\left(\rho_i\right)$ with $\rho_{AB} = \sum_i q_i \rho_i$ ({q_i, ρ_i} depend on Λ_1, ..., Λ_{n-1}, but not on Λ_n). By Pinsker's ineq. and convexity of x^2:

$$I(A:B_n|B_1,...,B_{n-1})_{\pi_\mu} \geq \frac{1}{2\ln 2}\left\| \Lambda_{A,j} \otimes \Lambda_{B_k}\left(\rho_{AB} - \sum_i q_i \rho_{i,A} \otimes \rho_{i,B_n}\right)\right\|_1^2$$

Thus: $$I(A:B_n|B_1,...,B_{n-1})_{\pi_\mu} \geq \frac{1}{2\ln 2}\max_{\Lambda_{B_k}} \underset{\mu}{E}\left\| \Lambda_{A,j} \otimes \Lambda_{B_k}\left(\rho_{AB} - \sum_i q_i \rho_{i,A} \otimes \rho_{i,B_n}\right)\right\|_1^2$$

Proof (part 2)

Similar tricks, but applied to multiparticle mutual information:

$$I(A_1 : \ldots : A_k) := S\left(\rho_{A_1 \ldots A_k} \,\|\, \rho_{A_1} \otimes \ldots \otimes \rho_{A_k}\right)$$

and using the useful inequality (Yang, Horodecki[3], Oppenheim, Song '07)

$$I(A_1 : \ldots : A_k) = I(A_1 : A_2) + I(A_3 : A_1 A_2) + \ldots + I(A_k : A_1 \ldots A_{k-1})$$

Part 2: de Finetti with no symmetry

Quantum de Finetti with no symmetry

Thm Let $p_{1,\ldots,n}$ be a prob. distribution over Σ^n. Let $p_{1,\ldots,n|j_1=a_1,\ldots,j_t=a_t}$ be the probability conditioned on observing $(j_1, \ldots, j_t) = (a_1, \ldots, a_t)$.

Then: $$\mathop{E}_{j_1,\ldots,j_t} \mathop{E}_{a_1,\ldots,a_t} \mathop{E}_{i_1,\ldots,i_k} \left\| p_{i_1\ldots i_k|j_1=a_1,\ldots,j_t=a_t} - p_{i_1|j_1=a_1,\ldots,j_t=a_t} \otimes \ldots \otimes p_{i_k|j_1=a_1,\ldots,j_t=a_t} \right\|_1$$

$$\leq \sqrt{\frac{2\ln(2)k(k-1)\log|\Sigma|}{t-1}}$$

Based on bound by (Raghavendra, Tan '11) (proposed in the context of bounding convergence of Lasserre hierarchy for certain CSP)

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Then: $\mathop{E}_{j_1,\ldots,j_t} \mathop{E}_{a_1,\ldots,a_t} \mathop{E}_{i_1,\ldots,i_k} \left\| p_{i_1\ldots i_k|j_1=a_1,\ldots,j_t=a_t} - p_{i_1|j_1=a_1,\ldots,j_t=a_t} \otimes \ldots \otimes p_{i_k|j_1=a_1,\ldots,j_t=a_t} \right\|_1$

$$\leq \sqrt{\frac{2\ln(2)k(k-1)\log|\Sigma|}{t-1}}$$

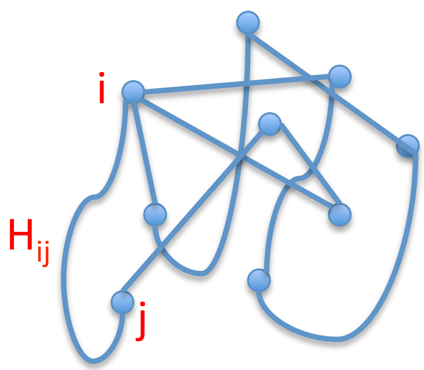
Quantum Version: $p_{1,\ldots,n} = \Lambda^{\otimes n}(\rho)$ for informationally complete POVM Λ. Same theorem (for trace norm) with error term

$$poly(d^k)\sqrt{\frac{k(k-1)\log d}{t-1}}$$

Quantum PCP

The PCP conjecture: There is a ε > 0 s.t. it's QMA-complete to determine whether the mean groundenergy of a 2-local Hamiltonian on *n* qubits is 0 or more than ε.

(Bravyi, diVincenzo Loss, Terhal 'XX) Equivalent to conjecture for O(1)-local Hamiltonians over q*d*its.



(Reminder) 2-Local Hamiltonian: $\sum_{i \sim_G j} H_{ij}$

Mean energy: $e_0 := E_0/|E|$

E_0: groundenergy

$|E|$: # of edges in G

By PCP theorem, the problem is at least NP-hard

Quantum PCP: Non-trivial approximation in NP

Cor: There is a constant $c > 0$ such that the following problem is NP-complete: Given a 2-Local Hamiltonian on n qudits with interaction graph of average degree deg, decide whether $e_0 = 0$ or $e_0 > cd\text{^}3/deg$.

Proof Outline:

NP-hardness follows from PCP theorem + Raz parallel repetition theorem

Interesting part: There is always a product state assignment with energy no bigger than $e_0 + cd^3/deg$ (proof by "conditioning de Finetti bound")

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This gives evidence against the PCP conjecture. At least it can only hold for $\varepsilon < cd^3/deg$. Note that classical PCP holds for graphs for which $\Omega(1) = \varepsilon >> poly(\Sigma)/deg = o(1)$.

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- The "improved error de Finetti Thm" is useful in (a) showing hardness of a constant error approximation of quantum value of games and (b) give a matching simulation to the BellQMA protocol for SAT of Chen and Drucker.
- The "de Finetti Thm with no symmetry" is useful in (a) giving a PTAS for estimating the energy of dense Hamiltonians and (b) give evidence against the quantum PCP conjecture.

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4. (Dis)prove Quantum PCP conjecture
5. Further develop the connection of BellQMA (optimization of Bell observables over separable states) to the hardness of Small Set Expansion and Unique Games problems:
   - Can we prove quasi-polynomial hardness of UG by this approach?
   - Can we prove convergence of O(log(n)) rounds of Lasserre hierarchy for Small Set Expansion by this approach?

Thanks!