A generalization of quantum Stein's Lemma

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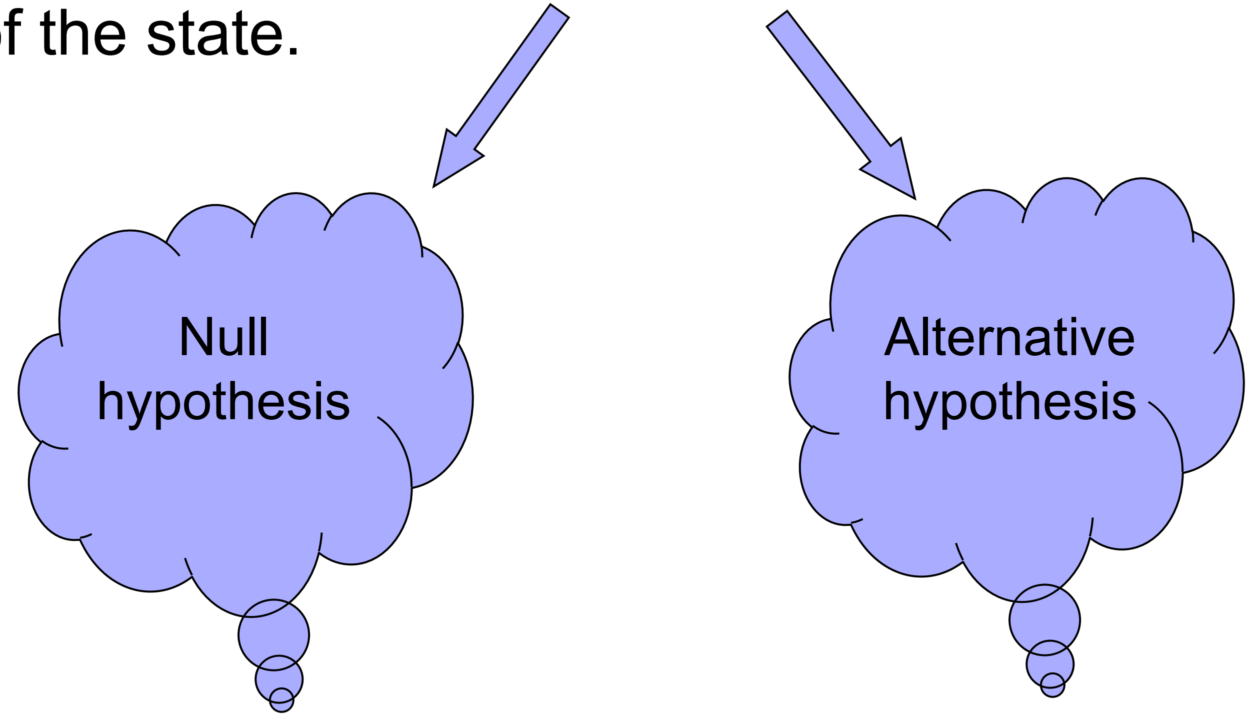
QIIC

(i.i.d.) Quantum Hypothesis Testing

- Given *n* copies of a quantum state, with the promise that it is described either by ρ or σ, determine the identity of the state.



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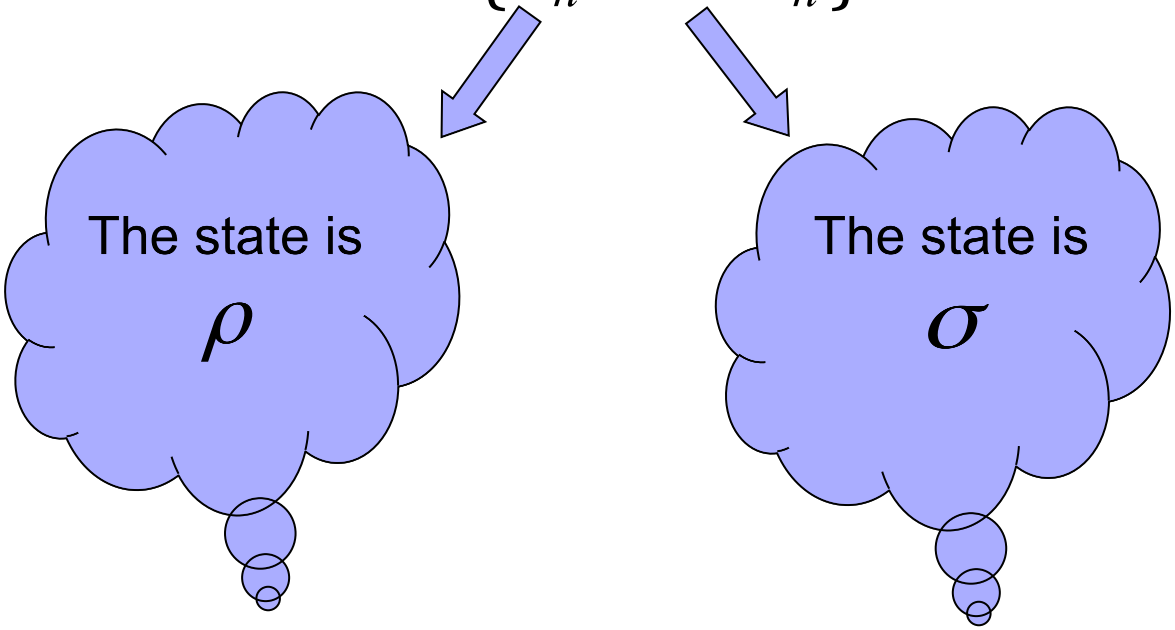
- Measure two outcome POVM $\{A_n, I - A_n\}$.



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- Given *n* copies of a quantum state, with the promise that it is described either by ρ or σ, determine the identity of the state

- Measure two outcome POVM $\{A_n, I - A_n\}$

- Error probabilities

  - Type I error: $$\alpha_n(A_n) := tr(\rho^{\otimes n}(I - A_n))$$

  - Type II error: $$\beta_n(A_n) := tr(\sigma^{\otimes n} A_n)$$

(i.i.d.) Quantum Hypothesis Testing

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Quantum Chernoff bound

(Audenaert, Nussbaum, Szkola, Verstraete 07)

$$\lim_{n\to\infty} \frac{\log r_n}{n} = \log \inf_{s\in[0,1]} tr(\rho^{1-s}\sigma^s)$$

Quantum Stein's Lemma

- Asymmetric hypothesis testing

$$r_n(\varepsilon) := \min_{0 \le A_n \le I} \beta_n(A_n) \ : \ \alpha_n(A_n) \le \varepsilon$$

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- Quantum Stein's Lemma
  (Hiai and Petz 91; Ogawa and Nagaoka 00)

$$\forall \varepsilon > 0, \ \lim_{n \to \infty} -\frac{\log r_n(\varepsilon)}{n} = S(\rho \| \sigma)$$

$$= tr(\rho(\log \rho - \log \sigma))$$

Quantum Stein's Lemma

- Most general setting

- Null hypothesis (null): $\rho_n \in \Omega_n \subset D(H^{\otimes n})$

- Alternative hypothesis (alt.): $\sigma_n \in \Xi_n \subset D(H^{\otimes n})$

Quantum Stein's Lemma

- Known results

  - $\left\{\pi^{\otimes n} : \pi \in \Omega \subset D(H)\right\}$ (null) versus $\sigma^{\otimes n}$ (alt.)

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$$tr(\sigma^{\otimes n} A_n)$$

$$\sup_{\pi \in \Omega} tr(\pi^{\otimes n}(I - A_n))$$

Quantum Stein's Lemma

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(Hayashi 00; Bjelakovic et al 04)

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  - *Ergodic* null hypothesis versus i.i.d. alternative hypothesis

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- *Ergodic* null hypothesis versus i.i.d. alternative hypothesis

(Hiai and Petz 91)

- General sequence of states: Information spectrum

(Han and Verdu 94; Nagaoka and Hayashi 07)

Quantum Stein's Lemma

- What about allowing the alternative hypothesis to be non-i.i.d. and to vary over a family of states?

Only ergodicity and related concepts seems not to be enough to define a rate for the decay of $\beta_n(A_n)$

(Shields 93)

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(Shields 93)

This talk: A setting where the optimal rate can be determined for varying correlated alternative hypothesis

A generalization of Quantum Stein's Lemma

- Consider the following two hypothesis

  - Null hypothesis: For every $n \in \mathbb{N}$ we have $\rho^{\otimes n}$

  - Alternative hypothesis: For every $n \in \mathbb{N}$ we have an unknown state $\omega_n \in \Omega_n \subset D(H^{\otimes n})$, where $\{\Omega_n\}_{n \in \mathbb{N}}$ satisfies:

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1. Each Ω_n is closed and convex
2. Each contains the maximally mixed state $I^{\otimes n} / \dim(H)^n$
3. If $\pi \in \Omega_{n+1}$, then $tr_j(\pi) \in \Omega_n \ \forall j \in \{1,\ldots,n+1\}$
4. If $\pi \in \Omega_n$ and $\sigma \in \Omega_m$, then $\pi \otimes \sigma \in \Omega_{n+m}$
5. If $\pi \in \Omega_n$, then $S_n(\pi) \in \Omega_n$

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$$S_n(*) = \sum_{\pi \in SYM(n)} P_\pi * P_\pi$$

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A generalization of Quantum Stein's Lemma

- **theorem**: Given $\{\Omega_n\}_{n\in N}$ satisfying properties 1-5 and $\rho \in D(H)$,

- (Direct Part) $\forall\ \varepsilon > 0$ there is a $\{A_n, I - A_n\}_{n\in N}$ s.t.

$$\lim_{n\to\infty} tr(A_n \rho^{\otimes n}) = 1$$

$$\forall\ \{\omega_n \in \Omega_n\}_{n\in N}, \quad tr(A_n \omega_n) \leq 2^{-n(E_\Omega^\infty(\rho) - \varepsilon)}$$

$$E_\Omega^\infty(\rho) = \lim_{n\to\infty} \min_{\sigma\in\Omega_n} \frac{S(\rho^{\otimes n} \| \sigma)}{n}$$

A generalization of Quantum Stein's Lemma

- **theorem**: Given $\{\Omega_n\}_{n\in N}$ satisfying properties 1-5 and $\rho \in D(H)$,

  - (Strong Converse) $\forall\ \varepsilon > 0,\ \{A_n, I - A_n\}_{n\in N}$ s.t.

$$\exists\ \{\omega_n \in \Omega_n\}_{n\in N} \quad s.t. \quad tr(A_n\omega_n) \leq 2^{-n(E_\Omega^\infty(\rho)+\varepsilon)}$$

$$\lim_{n\to\infty} tr(A_n\rho^{\otimes n}) = 0$$

A motivation: Entanglement theory

- We say $\sigma \in D(H = H_1 \otimes \ldots \otimes H_k)$ is separable if

$$\sigma = \sum_j p_j \sigma_j^1 \otimes \ldots \otimes \sigma_j^k$$

If it cannot be written in this form, it is *entangled*

- The sets of separable sates over $H^{\otimes n}$, $S(H^{\otimes n})$ satisfy properties 1-5

- The rate function of the theorem is a well-known entanglement measure, the *regularized relative entropy of entanglement* (Vedral and Plenio 98)

Regularized relative entropy of entanglement

- Given an entangled state $\rho \in D(H = H_1 \otimes \ldots \otimes H_k)$

$$E_R^{\infty}(\rho) = \lim_{n \to \infty} \min_{\sigma \in S(H^{\otimes n})} \frac{S(\rho^{\otimes n} \| \sigma)}{n}$$

(Vedral and Plenio 98)

- The theorem gives an *operational* interpretation to this measure as the optimal rate of discrimination of an entangled state to a arbitrary family of separable states

- More on the relative entropy of entanglement on Wednesday

Regularized relative entropy of entanglement

- **Cor:** For every entangled state $\rho \in D(H = H_1 \otimes ... \otimes H_k)$

$$E_R^{\infty}(\rho) > 0$$

Regularized relative entropy of entanglement

- Rate of conversion of two states by local operations and classical communication:

$$R(\rho \to \sigma) = \inf_{\{k_n\}} \left\{ \limsup_{n\to\infty} \frac{k_n}{n} : \lim_{n\to\infty} \left(\min_{\Lambda \in LOCC} \| \Lambda(\rho^{\otimes k_n}) - \sigma^{\otimes n} \|_1 \right) = 0 \right\}$$

- The corollary implies that if σ is entangled,

$$R(\rho \to \sigma) > 0$$

- The mathematical definition of entanglement is equal to the operational: multipartite bound entanglement is real

For bipartite systems see Yang, Horodecki, Horodecki, Synak-Radtke 05

Some elements of the proofs

- Asymptotic continuity: Let $E_{\Omega_n}(\rho) := \min_{\sigma \in \Omega_n} S(\rho \| \sigma), \;\; E_{\Omega}^{\infty}(\rho)$

$$\left| E(\rho) - E\ (\sigma) \right| \leq f(\| \rho - \sigma \|_1) n, \quad for \;\; \rho, \sigma \in D(H^{\otimes n})$$

(Horodecki and Synak-Radtke 05; Christandl 06)

- Non-lockability: Let $\rho = \sum_j p_j \rho_j$

$$\sum_j p_j E_{\Omega_n}(\rho_j) - h(p_j) \leq E_{\Omega_n}(\rho) \leq \sum_j p_j E_{\Omega_n}(\rho_j)$$

(Horodecki[3] and Oppenheim 05)

Some elements of the proofs

- **Lemma**: Let $\rho \in D(H)$ and $Y, \Delta \geq 0$ s.t. $\rho \leq Y + \Delta$

  Then $\exists \rho' \in D(H)$ s.t. $F(\rho', \rho) \geq 1 - tr(\Delta)$

  and $\rho' \leq Y$

  (Datta and Renner 08)

- **Lemma**: Let $\rho, \sigma \in D(H)$, $\forall \varepsilon > 0$

$$\lim_{n \to \infty} tr(\rho^{\otimes n} - 2^{(S(\rho\|\sigma)+\varepsilon)n} \sigma^{\otimes n})_+ = 0$$

(Ogawa and Nagaoka 00)

Some elements of the proofs

- Almost power states:

$$V(H^{\otimes n}, |\theta\rangle^{\otimes n-r}) := \left\{ P_\pi \left(|\theta\rangle^{\otimes n-r} \otimes |\psi_r\rangle \right) : \pi \in SYM(n), |\psi_r\rangle \in H^{\otimes r} \right\}$$

$$|\theta\rangle^{[\otimes, n, r]} = Sym(H^{\otimes n}) \cap span(V(H^{\otimes n}, |\theta\rangle^{\otimes n-r}))$$

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- **Exponential de Finetti theorem:** For any permutation-symmetric state $\rho_{n+k} \in D(H^{\otimes n+k})$ there exists a measure μ over $H \otimes H_E$ and states $|\psi_n^\theta\rangle \in |\theta\rangle^{[\otimes, n, r]}$ s.t.

$$\left\| tr_{1,\ldots,k}(\rho_{n+k}) - \int \mu(d|\theta\rangle) tr_E \left(|\psi_n^\theta\rangle\langle\psi_n^\theta| \right) \right\|_1 \leq n^{\dim(H)^2} 2^{-\frac{k(r+1)}{n+k}}$$

(Renner 05)

Elements of the proof

- (Proof sketch) We can write the statement of the theorem as

$$\lim_{n\to\infty} \lambda_n(\rho^{\otimes n}, 2^{yn}) = \begin{cases} 1, & y < E_\Omega^\infty(\rho) \\ 0, & y > E_\Omega^\infty(\rho) \end{cases}$$

$$\lambda_n(\rho^{\otimes n}, 2^{yn}) = \max_{0\leq A\leq I} tr(A\rho^{\otimes n}) : tr(A\omega) \leq 2^{-yn} \;\; \forall \sigma \in \Omega_n$$

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The dual formulation of the convex optimization above reads

$$\lambda_n(\rho^{\otimes n}, 2^{yn}) = \min_{\omega\in\Omega_n, b\in\Re} tr(\rho^{\otimes n} - 2^{bn}\omega)_+ + 2^{n(b-y)}$$

It is then clear that it suffices to prove

$$\lim_{n\to\infty} \min_{\omega\in\Omega_n} tr(\rho^{\otimes n} - 2^{yn}\omega)_+ = \begin{cases} 1, & y < E_{\Omega}^{\infty}(\rho) \\ 0, & y > E_{\Omega}^{\infty}(\rho) \end{cases}$$

Elements of the proof

- (Proof sketch) We first show that for every

$$\lim_{n\to\infty}\min_{\omega\in\Omega_n} tr(\rho^{\otimes n} - 2^{(E_\Omega^\infty(\rho)+\varepsilon)n}\omega)_+ = 0$$

Take n sufficiently large such that $\left|E_\Omega^\infty(\rho) - E_{\Omega_n}(\rho^{\otimes n})/n\right| \leq \varepsilon/2$

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Let $\omega_n \in \Omega_n$ be such that $E_{\Omega_n}(\rho^{\otimes n}) = S(\rho \| \omega_n)$

By the strong converse of quantum Stein's Lemma

$$\lim_{n\to\infty} tr(\rho^{\otimes nm} - 2^{(E_\Omega^\infty(\rho)+\varepsilon/2)nm}\omega_n^{\otimes m})_+ = 0$$

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- **Lemma**: Let $\rho, \sigma \in D(H)$, $\forall \varepsilon > 0$

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(Ogawa and Nagaoka 00)

Elements of the proof

- (Proof sketch) We now show

$$\lim_{n\to\infty} \min_{\omega\in\Omega_n} tr(\rho^{\otimes n} - 2^{(E_\Omega^\infty(\rho)-\varepsilon)n}\omega)_+ = \lambda > 0$$

Let $\omega_n \in \Omega_n$ be an optimal sequence in the eq. above

We can write $\rho^{\otimes n} \leq 2^{(E_\Omega^\infty(\rho)-\varepsilon)n}\omega_n + (\rho^{\otimes n} - 2^{(E_\Omega^\infty(\rho)-\varepsilon)n}\omega_n)_+$

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Assuming conversely that the limit is zero, we find

$$\rho_n \leq 2^{(E_\Omega^\infty(\rho)-\varepsilon)n}\omega_n, \ \left\|\rho_n - \rho^{\otimes n}\right\|_1 \to 0$$

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- (Proof sketch) We now show

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$$\rho_n \leq 2^{(E_\Omega^\infty(\rho)-\varepsilon)n}\omega_n, \quad \left\|\rho_n - \rho^{\otimes n}\right\|_1 \to 0$$

Then

$$E_\Omega^\infty(\rho) = \lim_{n\to\infty}\frac{E_{\Omega_n}(\rho_n)}{n} \leq E_\Omega^\infty(\rho) - \varepsilon$$

Elements of the proof

- (Proof sketch) We now show

$$\lim_{n\to\infty}\min_{\omega\in\Omega_n} tr(\rho^{\otimes n} - 2^{(E_\Omega^\infty(\rho)-\varepsilon)n}\omega)_+ = \lambda > 0$$

- Asymptotic continuity: Let $E_{\Omega_n}(\rho) := \min_{\sigma\in\Omega_n} S(\rho\|\sigma)$, $E_\Omega^\infty(\rho)$

$$\left|E(\rho) - E(\sigma)\right| \leq f(\|\rho-\sigma\|_1)n, \quad for \;\; \rho,\sigma \in D(H^{\otimes n})$$

(Horodecki and Synak-Radtke 05; Christandl 06)

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$$\rho_n \leq 2^{(E_\Omega^\infty(\rho)-\varepsilon)n}\omega_n, \;\; \left\|\rho_n - \rho^{\otimes n}\right\|_1 \to 0$$

Then

$$E_\Omega^\infty(\rho) = \lim_{n\to\infty}\frac{E_{\Omega_n}(\rho_n)}{n} \leq E_\Omega^\infty(\rho) - \varepsilon$$

Elements of the proof

- (Proof sketch) Finally we now show

$$\lim_{n\to\infty}\min_{\omega\in\Omega_n} tr(\rho^{\otimes n} - 2^{(E_\Omega^\infty(\rho)-\varepsilon)n}\omega)_+ = \lambda > 0$$

with $\lambda = 1$. Suppose conversely that $\lambda < 1$ .From

$$\rho^{\otimes n} \leq 2^{(E_\Omega^\infty(\rho)-\varepsilon)n}\omega_n + (\rho^{\otimes n} - 2^{(E_\Omega^\infty(\rho)-\varepsilon)n}\omega_n)_+$$

we can write

$$\rho_n \leq 2^{(E_\Omega^\infty(\rho)-\varepsilon)n}\omega_n, \;\; F(\rho_n, \rho^{\otimes n}) \geq \lambda$$

Note that we can take $\rho_n, \; \omega_n$ to be permutation-symmetric

Elements of the proof

- (Proof sketch) Define $\rho_{\alpha n} := tr_{1,\dots,\alpha n}(\rho_n),\ \omega_{\alpha n} := tr_{1,\dots,\alpha n}(\omega_n)$

  We have

$$\rho_{\alpha n} \leq 2^{(E_\Omega^\infty(\rho)-\varepsilon)n}\omega_{\alpha n},\ \ F(\rho_{\alpha n}, \rho^{\otimes(1-\alpha)n}) \geq \lambda$$

We can write

$$\int \mu(d|\theta\rangle)\pi_n^{|\theta\rangle} \leq 2^{(E_\Omega^\infty(\rho)-\varepsilon)n}\omega_{\alpha n} + X_n,\ \ \|X_n\|_1 \leq n^{-10d^2}$$

$$\pi_n^{|\theta\rangle} = tr_E\left(\left|\psi_{(1-\alpha)n}^{|\theta\rangle}\right\rangle\left\langle\psi_{(1-\alpha)n}^{|\theta\rangle}\right|\right),\ \ \left|\psi_{(1-\alpha)n}^{|\theta\rangle}\right\rangle \in |\theta\rangle^{[\otimes,(1-\alpha)n,11d^2\alpha^{-1}\log(n)]}$$

- **Exponential de Finetti theorem:** For any permutation-symmetric state $\rho_{n+k} \in D(H^{\otimes n+k})$ there exists a measure μ over $H \otimes H_E$ and states $|\psi_n^\theta\rangle \in |\theta\rangle^{\otimes, n, r}$ s.t.

$$\left\| tr_{1,\ldots,k}(\rho_{n+k}) - \int \mu(d|\theta\rangle) tr_E\left(|\psi_n^\theta\rangle\langle\psi_n^\theta|\right)\right\|_1 \leq n^{\dim(H)^2} 2^{-\frac{k(r+1)}{n+k}}$$

(Renner 05)

$$\int \mu(d|\theta\rangle)\pi_n^{|\theta\rangle} \leq 2^{(E_\Omega^\infty(\rho)-\varepsilon)n}\omega_{\alpha n} + X_n, \quad \|X_n\|_1 \leq n^{-10d^2}$$

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Elements of the proof

- (Proof sketch) Because $F(\rho_{\alpha n}, \rho^{\otimes(1-\alpha)n}) \geq \lambda$

$$\int_{B_{n^{-1/8}}(\rho)} \mu(d|\theta\rangle) = \Omega(1)$$

Elements of the proof

- (Proof sketch) Because $F(\rho_{\alpha n}, \rho^{\otimes(1-\alpha)n}) \geq \lambda$

$$\int_{B_{n^{-1/8}}(\rho)} \mu(d|\theta\rangle) = \Omega(1)$$

Therefore we can write

$$\int_{B_{n^{-1/8}}(\rho)} \mu'(d|\theta\rangle)\pi_n^{|\theta\rangle} \leq O(1)2^{(E_\Omega^\infty(\rho)-\varepsilon)n}\omega_{\alpha n} + O(1)X_n, \ \|X_n\|_1 \leq n^{-10d^2}$$

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and

$$\int_{B_{n^{-4}}(\rho')} \mu''(d|\theta\rangle)\pi_n^{|\theta\rangle} \leq n^{8d^2} 2^{(E_\Omega^\infty(\rho)-\varepsilon)n}\omega_{\alpha n} + n^{8d^2}X_n, \quad \|X_n\|_1 \leq n^{-10d^2}$$

with $\rho' \in B_{n^{-1/8}}(\rho)$

Elements of the proof

- (Proof sketch) Therefore

$$\pi_n^{|\theta'\rangle} \leq n^{8d^2} 2^{(E_\Omega^\infty(\rho)-\varepsilon)n} \omega_{\alpha n} + X_n', \ \ \|X_n\|_1 \leq n^{-1}$$

with $|\theta'\rangle \quad s.t. \ \ tr_E(|\theta'\rangle\langle\theta'|) = \rho'$

Finally

$$(1-\alpha)E_\Omega^\infty(\rho) = \limsup_{n\to\infty} \frac{E_{\Omega_n}(\pi_n^{|\theta'\rangle})}{n} \leq E_\Omega^\infty(\rho) - \varepsilon$$

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and

$$\int_{B_{n^{-4}}(\rho')} \mu''(d|\theta\rangle)\pi_n^{|\theta\rangle} \leq n^{8d^2} 2^{(E_\Omega^\infty(\rho)-\varepsilon)n}\omega_{\alpha n} + n^{8d^2} X_n, \ \|X_n\|_1 \leq n^{-10d^2}$$

with $\rho' \in B_{n^{-1/8}}(\rho)$