Quantum Hamiltonian Complexity

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Based on joint work with A. Harrow and M. Horodecki

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Quantum is Hard

Use of DoE supercomputers by area
(from a talk by Alán Aspuru-Guzik)

More than **33%** of DoE supercomputer power is devoted to simulating quantum physics

Can we get a better handle on this **simulation** problem?
Quantum Information Science

...gives new approaches

1. Quantum computer and quantum simulators
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2. Better classical algorithms for simulating quantum systems
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Quantum Many-Body Systems

Quantum Hamiltonian

\[ H = \sum_{i=1}^{m} H_i \in \left(C^d \right)^{\otimes n} \]

Interested in computing properties such as minimum energy, correlations functions at zero and finite temperature, dynamical properties, ...
Quantum Hamiltonian Complexity

...analyzes quantum many-body physics through the computational lens

1. Relevant for condensed matter physics, quantum chemistry, statistical mechanics, quantum information

2. Natural generalization of the study of constraint satisfaction problems in theoretical computer science
Constraint Satisfaction Problems vs Local Hamiltonians

$k$-arity CSP:

Variables $\{x_1, \ldots, x_n\}$, alphabet $\Sigma$

Constraints: $c_j : \Sigma^k \to \{0,1\}$

Assignment: $\sigma : [n] \to \Sigma$

Unsat := $\min_{\sigma} \sum_j c_j(\sigma(x_{j_1}), \ldots, \sigma(x_{j_k}))$
**Constraint Satisfaction Problems vs Local Hamiltonians**

**k-arity CSP:**

- **Variables**: \( \{x_1, \ldots, x_n\} \), alphabet \( \Sigma \)
- **Constraints**: \( c_j : \Sigma^k \rightarrow \{0,1\} \)
- **Assignment**: \( \sigma : [n] \rightarrow \Sigma \)
- **Unsat** := \( \min_{\sigma} \sum_j c_j(\sigma(x_{j_1}),\ldots,\sigma(x_{j_k})) \)

**k-local Hamiltonian \( H \):**

- **n qudits in** \( (C^d)^\otimes n \)
- **Constraints**: \( H_j \in \text{Her}((C^d)^\otimes k) \)
- **qUnsat** := \( E_0 \left( \sum_j H_j \right) \)
- **\( E_0 \)**: min eigenvalue
C. vs Q. Optimal Assignments

Finding optimal assignment of CSPs can be hard
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Finding optimal assignment of CSPs can be hard

Finding optimal assignment of quantum CSPs can be even harder

(BCS Hamiltonian groundstate, Laughlin states for FQHE,...)
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Main difference: Optimal Assignment can be a highly entangled state (unit vector in $(C^d)^\otimes n$)
Optimal Assignments: Entangled States

Non-entangled state: \[ \left( a_1 |0\rangle + b_1 |1\rangle \right) \otimes \ldots \otimes \left( a_n |0\rangle + b_n |1\rangle \right) \]
e.g. \[ |\uparrow\rangle \otimes \ldots \otimes |\uparrow\rangle \]

Entangled states: \[ \sum_{i_1,\ldots,i_n} c_{i_1,\ldots,i_n} |i_1,\ldots,i_n\rangle \]
e.g. \[ \left( |\uparrow\rangle |\downarrow\rangle - |\downarrow\rangle |\uparrow\rangle \right) / \sqrt{2} \]

To describe a general entangled state of \( n \) spins requires \( \exp(O(n)) \) bits
How Entangled?

Given bipartite entangled state \( |\psi\rangle_{AB} \in C^n \otimes C^m \)

the reduced state on A is mixed: \( \rho_A \geq 0, \quad tr(\rho_A) = 1 \)

The more mixed \( \rho_A \), the more entangled \( \psi_{AB} \):

Quantitatively: \( E(\psi_{AB}) := S(\rho_A) = -tr(\rho_A \log \rho_A) \)

Is there a relation between the amount of entanglement in the ground-state and the computational complexity of the model?
Outline

• Quantum PCP Conjecture
  What is it?
  Limitations to qPCP
  New algorithms

• Area Law
  What is it?
  Area Law from Decay of Correlations
  Proof by Quantum Shannon Theory
**NP ≠ Non-Polynomial**

NP is the class of problems for which one can check the correctness of a potential efficiently (in polynomial time).

E.g. **Factoring**: Given N, find a number that divides it, $N = m \times q$.

E.g. **Graph Coloring**: Given a graph and k colors, color the graph such that no two neighboring vertices have the same color.

![3-coloring diagram]
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**The million dollars question:**

**Is P = NP?**

3-coloring
A problem is **NP-hard** if any other problem in NP can be reduced to it in polynomial time.

E.g. **3-SAT**: CSP with binary variables \( x_1, \ldots, x_n \) and constraints \( \{ C_i \} \), \( C_i = x_{i_1} \land \overline{x}_{i_2} \land x_{i_3} \)

**Cook-Levin Theorem**: 3-SAT is NP-hard
NP-hardness

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E.g. There is an efficient mapping between graphs and 3-SAT formulas such that given a graph $G$ and the associated 3-SAT formula $S$

$G$ is 3-colorable $\iff$ $S$ is satisfiable
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\[ G \text{ is 3-colorable } \iff \text{S is satisfiable} \]

**NP-complete**: NP-hard + inside NP
Complexity of qCSP

Since computing the ground-energy of local Hamiltonians is a generalization of solving CSPs, the problem is at least NP-hard.

Is it in NP? Or is it harder?

The fact that the optimal assignment is a highly entangled state might make things harder...
The Local Hamiltonian Problem

Problem
Given a local Hamiltonian $H$, decide if $E_0(H) = 0$ or $E_0(H) > \Delta$

$E_0(H)$ : minimum eigenvalue of $H$

Thm (Kitaev ‘99) The local Hamiltonian problem is QMA-complete for $\Delta = 1/\text{poly}(n)$
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(Analogue Cook-Levin thm)

QMA is the quantum analogue of NP, where the proof and the computation are quantum.

Input

Witness

\[ \begin{align*}
U_1 & \quad \cdots \quad U_4 & \quad U_5 \\
U_2 & & & \\
\end{align*} \]
The meaning of it

It’s widely believed QMA ≠ NP

Thus, there is generally no \textit{efficient} classical description of groundstates of local Hamiltonians

Even very simple models are QMA-complete
E.g.
(Aharonov, Irani, Gottesman, Kempe ‘07) 1D models

“1D systems as hard as the general case”
The meaning of it

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“1D systems as hard as the general case”

What’s the role of the accuracy \(\Delta\) on the hardness?

... But first what happens classically?
PCP Theorem (Arora et al ’98, Dinur ‘07): There is a $\varepsilon > 0$ s.t. it’s NP-complete to determine whether for a CSP with $m$ constraints, $\text{Unsat} = 0$ or $\text{Unsat} > \varepsilon m$
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- NP-hard even for $\Delta = \Omega(m)$

- Equivalent to the existence of \textbf{P}robabilistically \textbf{C}heckable \textbf{P}roofs for NP.

- Central tool in the theory of \textbf{hardness of approximation} (optimal threshold for 3-SAT ($7/8$-factor), max-clique ($n^{1-\varepsilon}$-factor))
Quantum PCP?

The qPCP conjecture: There is $\varepsilon > 0$ s.t. the following problem is QMA-complete: Given 2-local Hamiltonian $H$ with $m$ local terms determine whether

(i) $E_0(H)=0$ or

(ii) $E_0(H) > \varepsilon m$. 

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- Equivalent to estimating mean groundenergy to constant accuracy ($e_o(H) := E_0(H)/m$)

- Related to estimating energy at constant temperature

- At least NP-hard (by PCP Thm) and in QMA
Quantum PCP?
Previous Work and Obstructions

(Aharonov, Arad, Landau, Vazirani ‘08)
Quantum version of 1 of 3 parts of Dinur’s proof of the PCP thm (gap amplification)

But: The other two parts (alphabet and degree reductions) involve massive copying of information; not clear how to do it with a highly entangled assignment
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(Bravyi, Vyalyi ’03; Arad ’10; Hastings ’12; Freedman, Hastings ’13; Aharonov, Eldar ’13, ...)
No-go for large class of commuting Hamiltonians and almost commuting Hamiltonians

But: Commuting case might always be in NP
Going Forward

• Can we understand why got stuck in quantizing the classical proof?
• Can we prove partial no-go beyond commuting case?

Yes, by considering the simplest possible reduction from quantum Hamiltonians to CSPs.
Mean-Field...

...consists in approximating groundstate by a product state $|\psi_1\rangle \otimes \ldots \otimes |\psi_n\rangle$

$$\max_{\psi_1,\ldots,\psi_n} \sum_j \langle \psi_1,\ldots,\psi_n | H_j | \psi_1,\ldots,\psi_n \rangle$$ is a CSP

It’s a mapping from quantum Hamiltonians to CSPs

Successful heuristic in Quantum Chemistry (Hartree-Fock)
Condensed matter (e.g. BCS theory)

Folklore:
Mean-Field good when Many-particle interactions
Low entanglement in state
(B., Harrow ‘12) Let $H$ be a 2-local Hamiltonian on qudits with interaction graph $G(V, E)$ and $|E|$ local terms.
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$m < O(\log(n))$
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Let $\{X_i\}$ be a partition of the sites with each $X_i$ having $m$ sites.

$E_i$ : expectation over $X_i$
$\text{deg}(G)$ : degree of $G$
$\Phi(X_i)$ : expansion of $X_i$
$S(X_i)$ : entropy of groundstate in $X_i$

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$$\frac{1}{|E|} \langle \psi_1, ..., \psi_m | H | \psi_1, ..., \psi_m \rangle \leq e_0(H) + \Omega \left( d^6 E_i \Phi(X_i) \frac{1}{\text{deg}(G)} E_i \frac{S(X_i)}{m} \right)^{1/8}$$

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**Approximation in terms of 3 parameters:**

1. Average expansion
2. Degree interaction graph
3. Average entanglement groundstate
Approximation in terms of average expansion

\[ \frac{1}{|E|} \langle \psi_1, \ldots, \psi_m | H | \psi_1, \ldots, \psi_m \rangle \leq e_0(H) + \Omega \left( d^6 E_i \Phi(X_i) \frac{1}{\text{deg}(G)} \frac{S(X_i)}{m} \right)^{1/8} \]

Average Expansion: \[ E_i \Phi(X_i) = E_i \Pr_{(u,v) \in E} \left( v \notin X_i \mid u \in X_i \right) \]

Well known fact: ‘s divide and conquer

Potential hard instances must be based on expanding graphs

\[ m < O(\log(n)) \]
Approximation in terms of degree

\[
\frac{1}{|E|} \left\langle \psi_1, \ldots, \psi_m \middle| H \middle| \psi_1, \ldots, \psi_m \right\rangle \leq e_0(H) + \Omega \left( d^6 E_i \Phi(X_i) \frac{1}{\deg(G)} E_i \frac{S(X_i)}{m} \right)^{1/8}
\]

No classical analogue:

(PCP + parallel repetition) For all $\alpha, \beta, \gamma > 0$ it’s NP-complete to determine whether a CSP $C$ is s.t.

\[
\text{Unsat} = 0 \text{ or } \text{Unsat} > \alpha \Sigma^\beta / \deg(G)^\gamma
\]

Parallel repetition: $C \rightarrow C'$

i. $\deg(G') = \deg(G)^k$

ii. $\Sigma' = \Sigma^k$

ii. $\text{Unsat}(G') > \text{Unsat}(G)$

(Raz ‘00) even showed $\text{Unsat}(G')$ approaches 1 exponentially fast
Approximation in terms of degree

\[ \frac{1}{|E|} \langle \psi_1, \ldots, \psi_m | H | \psi_1, \ldots, \psi_m \rangle \leq e_0(H) + \Omega \left( d^6 E_i \Phi(X_i) \frac{1}{\deg(G)} E_i \frac{S(X_i)}{m} \right)^{1/8} \]

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Q. Parallel repetition: \( H \rightarrow H' \)

   i. \( \deg(H') = \deg(H)^k \)
   ii. \( d' = d^k \)
   iii. \( e_0(H') > e_0(H) \)
Approximation in terms of degree

\[
\frac{1}{|E|} \langle \psi_1, \ldots, \psi_m | H | \psi_1, \ldots, \psi_m \rangle \leq e_0(H) + \Omega \left( d^6 E_i \Phi(X_i) \frac{1}{\text{deg}(G)} E_i \frac{S(X_i)}{m} \right)^{1/8}
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**(PCP + parallel repetition)** For all \( \alpha, \beta, \gamma > 0 \) it’s NP-complete to determine whether a CSP \( C \) is s.t.

Unsat = 0 or Unsat > \( \alpha \Sigma^\beta / \text{deg}(G)^\gamma \)

**Contrast:** It’s in NP determine whether a Hamiltonian \( H \) is s.t

\( e_0(H) = 0 \) or \( e_0(H) > \alpha d^{3/4} / \text{deg}(G)^{1/8} \)

Quantum generalizations of PCP *and* parallel repetition *cannot* both be true (assuming QMA not in NP)
Approximation in terms of degree

\[
\frac{1}{|E|} \langle \psi_1, \ldots, \psi_m | H | \psi_1, \ldots, \psi_m \rangle \leq e_0(H) + \Omega \left( d^6 E_i \Phi(X_i) \frac{1}{\text{deg}(G)} E_i \frac{S(X_i)}{m} \right)^{1/8}
\]

Bound: \( \Phi_G < \frac{1}{2} - \Omega(1/\text{deg}) \) implies

Highly expanding graphs (\( \Phi_G \rightarrow 1/2 \)) are not hard instances

Obs: Restricted to 2-local models

(Aharonov, Eldar '13) k-local, commuting models
Approximation in terms of degree

...shows mean field becomes exact in high dim

1-D

2-D

3-D

∞-D

Rigorous justification to folklore in condensed matter physics
Approximation in terms of average entanglement

\[ \frac{1}{|E|} \langle \psi_1, \ldots, \psi_m | H | \psi_1, \ldots, \psi_m \rangle \leq e_0(H) + \Omega \left( d^6 E_i \Phi(X_i) \frac{1}{\text{deg}(G)} E_i \frac{S(X_i)}{m} \right)^{1/8} \]

Mean field works well if entanglement of groundstate satisfies a subvolume law:

\[ E_i \frac{S(X_i)}{m} = o(1) \]

Connection of amount of entanglement in groundstate and computational complexity of the model

\[ m < O(\log(n)) \]
Approximation in terms of average entanglement

\[
\frac{1}{|E|} \langle \psi_1, \ldots, \psi_m | H | \psi_1, \ldots, \psi_m \rangle \leq e_0(H) + \Omega \left( d^6 E_i \Phi(X_i) \frac{1}{\text{deg}(G)} E_i \frac{S(X_i)}{m} \right)^{1/8}
\]

Systems with low entanglement are expected to be easy.

So far only precise in 1D:

Area law for entanglement \(\rightarrow\) MPS description

Here:

Good: arbitrary lattice, only subvolume law

Bad: Only mean energy approximated well
New Classical Algorithms for Quantum Hamiltonians

Following same approach we also obtain polynomial time algorithms for approximating the groundstate energy of

1. Planar Hamiltonians, improving on (Bansal, Bravyi, Terhal ‘07)
2. Dense Hamiltonians, improving on (Gharibian, Kempe ‘10)
3. Hamiltonians on graphs with low threshold rank, building on (Barak, Raghavendranda, Steurer ‘10)

In all cases we prove that a product state does a good job and use efficient algorithms for CSPs.
Proof Idea: Monogamy of Entanglement

Cannot be highly entangled with too many neighbors

Entropy quantifies how entangled it can be

Proof uses information-theoretic techniques to make this intuition precise

Inspired by classical information-theoretic ideas for bounding convergence of SoS hierarchy for CSPs
(Tan, Raghavendra ‘10, Barak, Raghavendra, Steurer ‘10)
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• Quantum PCP Conjecture
  What is it?
  Limitations to qPCP
  New algorithms

• Area Law
  What is it?
  Area Law from Decay of Correlations
  Proof by Quantum Information Theory
Area Law

How complex are groundstates of local models? Given $|\psi_0\rangle$, how much entanglement does it have?

Area law means the entanglement is proportional to the perimeter of $A$ only (stepping stone to many approximation schemes based on tensor network states (PEPS, MERA, etc))
Why Area Law?

The intuition comes from the fact that correlations decay exponentially in groundstates of non-critical models

(Hastings ’04, Nachtergaele, Sims ‘06, Koma ‘06)

Spectral Gap: \( \Delta(H) := E_1 - E_0 \)

Non critical Hamiltonians are gapped
Condensed (matter) version of Area Law from Exponential Decay of Correlations

- Finite correlation length implies correlations are short ranged
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Condensed (matter) version of Area Law from Exponential Decay of Correlations

- Finite correlation length implies correlations are short ranged
Finite correlation length implies correlations are short ranged

- A is only entangled with B at the boundary: area law
Condensed (matter) version of Area Law from Exponential Decay of Correlations

- Is the intuition correct?
- Can we make it precise?

- Finite correlation length implies correlations are short ranged
- A is only entangled with B at the boundary: area law
Exponential Decay of Correlations

Let $|\psi\rangle_{1,...,n} \in (C^2)^\otimes n$ be a $n$-qubit quantum state

Correlation Function:

$$Cor(X : Z) := \max_{\|M\|,\|N\|\leq 1} \left|tr\left((M \otimes N)(\rho_{XZ} - \rho_X \otimes \rho_Z)\right)\right|$$
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$\text{Cor}(X : Z) := \max_{\|M\|,\|N\| \leq 1} \left| \text{tr} \left( (M \otimes N)(\rho_{XZ} - \rho_X \otimes \rho_Z) \right) \right|$

$= \max_{\|M\|,\|N\| \leq 1} \left| \langle M_X N_Z \rangle_{\psi} - \langle M_X \rangle_{\psi} \langle N_Z \rangle_{\psi} \right|$
Exponential Decay of Correlations

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\[ C^2 \]

\[ X \quad Y \quad Z \]

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Exponential Decay of Correlations: There is $\xi > 0$ s.t. for all cuts $X, Y, Z$ with $|Y| = 1$

\[ Cor(X : Z) \leq 2^{-l/\xi} \]
Exponential Decay of Correlations: There is $\xi > 0$ s.t. for all cuts $X, Y, Z$ with $|Y| = l$

$$Cor(X : Z) \leq 2^{-l/\xi}$$

$\xi$: correlation length
Area Law in 1D

Let $|\psi\rangle_{1,...,n} \in (C^2)^{\otimes n}$ be a $n$-qubit quantum state.

Entanglement Entropy: $E\left(|\psi_{XY}\rangle\right) := S(\rho_X)$

Area Law: For all partitions of the chain $(X, Y)$

$S(\rho_X) \leq \text{const}$
Area Law in 1D

**Area Law:** For all partitions of the chain \((X, Y)\)

\[
S(\rho_X) \leq \text{const}
\]

For the majority of quantum states: 

\[
S(\rho_X) \approx \text{size}(X) = r
\]

Area Law puts severe constraints on the amount of entanglement of the state.
## States that satisfy Area Law

**Intuition** - based on concrete examples (XY model, harmonic systems, etc.) and general non-rigorous arguments:

<table>
<thead>
<tr>
<th>Model</th>
<th>Spectral Gap</th>
<th>Area Law</th>
</tr>
</thead>
<tbody>
<tr>
<td>Non-critical</td>
<td>Gapped</td>
<td>$S(X) \leq O(Area(X))$</td>
</tr>
<tr>
<td>Critical</td>
<td>Non-gapped</td>
<td>$S(X) \leq O(Area(X)\log(n))$</td>
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## States that satisfy Area Law

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<tr>
<th>Reference</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Aharonov et al ’07; Irani ’09, Irani, Gottesman ‘09)</td>
<td>Groundstates 1D Ham. with volume law ( S(X) \geq \Omega(\text{vol}(X)) )</td>
</tr>
<tr>
<td>(Hastings ‘07)</td>
<td>Groundstates 1D \textit{gapped} local Ham. ( S(X) \leq 2^{O(1/\Delta)} )</td>
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<td>(Wolf, Verstraete, Hastings, Cirac ‘07)</td>
<td>Thermal states of local Ham. ( I(X:Y) \leq O(\text{Area}(X)/\beta) )</td>
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<tr>
<td>(Arad, Kitaev, Landau, Vazirani ‘12)</td>
<td>Groundstates 1D \textit{gapped} local Ham. ( S(X) \leq O(1/\Delta) )</td>
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</table>
Area Law and MPS

Matrix Product State (MPS):

\[ |\psi\rangle_{1,...,n} = \sum_{i_1=1}^{2} \ldots \sum_{i_n=1}^{2} \text{tr}\left( A_{i_1}^{[1]} \ldots A_{i_n}^{[n]} \right) |i_1,\ldots,i_n\rangle, \quad A_j^{[l]} \in \text{Mat}(D,D) \]

\( D \): bond dimension

- Only \( nD^2 \) parameters.
- Local expectation values computed in \( \text{poly}(D, n) \) time
- Variational class of states for powerful DMRG

In 1D: Area Law  \implies\  State has an efficient classical description MPS with \( D = \text{poly}(n) \)

(Vidal ‘03, Verstraete, Cirac ‘05, Schuch, Wolf, Verstraete, Cirac ‘07, Hastings ‘07)
Area Law in 1D

Let $\ket{\psi}_{1,\ldots,n} \in (C^2)^\otimes n$ be a $n$-qubit quantum state

Entanglement Entropy: $E\left(\ket{\psi_{XY}}\right) := S(\rho_X)$

Area Law: For all cuts of the chain $(X, Y)$, with $X = [1, r]$, $Y = [r+1, n]$,

$$S(\rho_X) \leq \text{const}$$
Area Law vs. Decay of Correlations

Exponential Decay of Correlations suggests Area Law
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Exponential Decay of Correlations suggests Area Law:

\[ I = O(\xi) \]

\[ |\psi\rangle_{XYZ} \]

\( \xi \)-EDC implies

\[ \rho_{XZ} \approx 2^{-l/\xi} \rho_X \otimes \rho_Z \]
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\[ X \quad Y \quad Z \]

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which implies

\[ |\psi\rangle_{XYZ} \approx 2^{-\frac{l}{\xi}} \left( U_{Y_1Y_2 \rightarrow Y} \otimes I_{XZ} \right) \left| \pi \right\rangle_{XY_1} \left| \nu \right\rangle_{Y_2Z} \] (by Uhlmann’s theorem)

\(X\) is only entangled with \(Y\)!
Area Law vs. Decay of Correlations

Exponential Decay of Correlations suggests Area Law:

$$I = O(\xi)$$

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X is only entangled with Y! Alas, the argument is wrong...

Uhlmann’s thm require 1-norm:

$$\left\| \rho_{AC} - \rho_A \otimes \rho_C \right\|_1 = 2 \max_{0 < M < I} \text{tr} \left( M \left( \rho_{AC} - \rho_A \otimes \rho_C \right) \right)$$
Area Law vs. Decay of Correlations

Exponential Decay of Correlations suggests Area Law:

\[ I = O(\xi) \]

\[ |\psi\rangle_{XYZ} \]

\[ X \quad Y \quad Z \]

\[ \xi\text{-EDC implies } \rho_{XZ} \approx 2^{-l/\xi} \rho_X \otimes \rho_Z \quad \text{which implies} \]

\[ |\psi\rangle_{XYZ} \approx 2^{-l/\xi} \left( U_{Y_1Y_2 \rightarrow Y} \otimes I_{XZ} \right) |\pi\rangle_{XY_1} |\nu\rangle_{Y_2Z} \quad \text{(by Uhlmann’s theorem)} \]

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Data Hiding States

Well distinguishable globally, bur poorly distinguishable locally

(DisVincenzo, Hayden, Leung, Terhal ’02)

Ex. 1 Antisymmetric Werner state \( \omega_{AB} = (I - F)/(d^2-d) \)

\[
Cor(A : B) \leq 1 / d, \quad \| \omega_{AB} - \omega_A \otimes \omega_B \|_1 \approx 1 / 2
\]

Ex. 2 Random state \( |\psi\rangle_{XYZ} \) with \( |X|=|Z| \) and \( |Y|=l \)

\[
Cor(X : Y) \leq 2^{-\Omega(l)}, \quad S(X) \approx (n - l) / 2
\]
What data hiding implies?

1. Intuitive explanation is flawed
What data hiding implies?

1. Intuitive explanation is **flawed**

2. **No-Go** for area law from exponential decaying correlations? So far believed to be so (by QI people)
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3. Cop out: data hiding states are unnatural; “physical” states are well behaved.
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3. **Cop out**: data hiding states are unnatural; “physical” states are well behaved.

4. We fixed a partition; **EDC** gives us more...

5. It’s an interesting quantum information problem: How strong is data hiding in quantum states?
**Exponential Decaying Correlations Imply Area Law**

*Thm 1 (B., Horodecki ‘12)* If $|\psi\rangle_{1,\ldots,n}$ has $\xi$-EDC then for every $X$,

$$S(X) \leq 2^{O(\xi \log(\xi))}$$
(Cor. Thm 1) If $|\psi\rangle_{1,...,n}$ has $\xi$-EDC then for every $\varepsilon>0$ there is MPS $|\psi_\varepsilon\rangle$ with poly(n, 1/\varepsilon) bound dim. s.t.

$$\left|\langle\psi|\psi_\varepsilon\rangle\right|\geq 1-\varepsilon$$

States with exponential decaying correlations are *simple* in a precise sense
Correlations in Q. Computation

What kind of correlations are necessary for exponential speed-ups?

1. (Vidal ‘03) Must exist $t$ and $X = [1, r]$ s.t. $S^\varepsilon_{\text{max}} (\rho_{t,X}) \geq n^\delta$
Correlations in Q. Computation

What kind of correlations are necessary for exponential speed-ups?

1. *(Vidal ‘03)* Must exist $t$ and $X = [1,r]$ s.t. $S^{\varepsilon}_{\text{max}} (\rho_{t,X}) \geq n^\delta$

2. *(Cor. Thm 1)* At some time step state must have long range correlations (at least algebraically decaying)
   - Quantum Computing happens in “critical phase”
   - Cannot hide information everywhere
Random States Have EDC?

$|\psi\rangle_{XYZ}$: Drawn from Haar measure

w.h.p, if size($X$) $\approx$ size($Z$): $\text{cor}(X : Z) \leq 2^{-\Omega(l)}$

and $S(X) \approx S(Z) \approx (n - l) / 2$

Small correlations in a fixed partition do not imply area law.
Random States Have EDC?

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Small correlations in a fixed partition do not imply area law.

But we can move the partition freely...
Random States Have Big Correl.

$\left| \psi \right\rangle_{XYZ}$ : Drawn from Haar measure

Let $\text{size}(XY) < \text{size}(Z)$. W.h.p. $\left\| \rho_{XY} - \tau_X \otimes \tau_Y \right\|_1 \leq 2^{-\Omega(n)}$, $\tau_X := \frac{I}{|X|}$

$X$ is decoupled from $Y$. 
Random States Have Big Correl.

Let \( \text{size}(XY) < \text{size}(Z) \). W.h.p. \[
\| \rho_{XY} - \tau_X \otimes \tau_Y \|_1 \leq 2^{-\Omega(n)}, \quad \tau_X := \frac{I}{|X|}
\]

\( X \) is decoupled from \( Y \).

Extensive entropy, but also large correlations:
Random States Have Big Correl.

\[ |\psi\rangle_{XYZ} : \text{Drawn from Haar measure} \]

Let size(XY) < size(Z). W.h.p. \[ \|\rho_{XY} - \tau_X \otimes \tau_Y\|_1 \leq 2^{-\Omega(n)}, \quad \tau_X := \frac{I}{|X|} \]

\( X \) is decoupled from \( Y \).

Extensive entropy, but also large correlations:

\[ U_{Z \rightarrow Z_1Z_2} |\psi\rangle_{XYZ} \approx |\Phi\rangle_{XZ_1} \otimes |\Phi\rangle_{YZ_2} \]

(\( U \)-Hmann’s theorem)

\[ |\Phi\rangle_{XZ_1} : \text{Maximally entangled state between } XZ_1. \]
Random States Have Big Correl.

Let size(XY) < size(Z). W.h.p. \( \| \rho_{XY} - \tau_X \otimes \tau_Y \|_1 \leq 2^{-\Omega(n)} \), \( \tau_X := \frac{I}{|X|} \)

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**Extensive entropy, but also large correlations:**

\[ U_{Z \rightarrow Z_1 Z_2} \left| \psi \right\rangle_{XYZ} \approx \left| \Phi \right\rangle_{XZ_1} \otimes \left| \Phi \right\rangle_{YZ_2} \]

(Uhlmann’s theorem)

\( \left| \Phi \right\rangle_{XZ_1} \) : Maximally entangled state between \( XZ_1 \).

\[ \text{Cor}(X:Z) \geq \text{Cor}(X:Z_1) = \Omega(1) >> 2^{-\Omega(n)} : \text{long-range correlations!} \]
Random States Have Big Correl.

It was thought random states were counterexamples to area law from EDC.

Not true; reason hints at the idea of the general proof:

We show large entropy leads to large correlations by choosing a random measurement that decouples $A$ and $B$.

$X$ is decoupled from $Y$.

$\sum_{i} I_{X_i} \geq \sum_{i} I_{X_i}^{(1)}$ also large correlations:

$\rho_{XZ1} = \Phi_{XZ1} \otimes \Phi_{YZ2}$

(Uhlmann’s theorem)

$\Cor(X:Z) \geq \Cor(X:Z_1) = \Omega(1) > 2^{-\Omega(n)}$ : long-range correlations!
The ingredients

We need to analyse decoupling and state merging in a single copy of a state. For that we use **single-shot information theory** (Renner et al ‘03, ...)

<table>
<thead>
<tr>
<th>State Merging</th>
<th>Single-Shot State Merging</th>
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<tbody>
<tr>
<td></td>
<td>(Dupuis, Berta, Wullschleger, Renner ‘10)</td>
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<tr>
<td></td>
<td>+ New bound on correlations by random measurements</td>
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<th>Saturation max- Mutual Info.</th>
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<td>Proof much more involved; based on</td>
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<tr>
<td></td>
<td>- Quantum substate theorem,</td>
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<td>- Quantum equipartition property,</td>
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<td></td>
<td>- Min- and Max-Entropies Calculus</td>
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<td></td>
<td>- EDC Assumption</td>
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Conclusions

• Quantum Hamiltonian Complexity studies quantum many-body physics through the computational lens.

• Two major open problems there are (i) the existence of a quantum PCP theorem and (ii) to prove area laws.

• Both are concerned with understanding better entanglement in groundstates. Quantum information theory is a powerful tool.
Thank you!