Quantum Physics of Information

Fernando G.S.L. Brandão

Caltech

PMA seminar, Caltech, 2016
Quantum is Hard

Use of DoE supercomputers by area

Large part of DoE supercomputer power is devoted to simulating quantum physics

Can we get a better handle on this simulation problem?
"trying to find a computer simulation of physics seems to me to be an excellent program to follow out (...) Nature isn’t classical, dammit, and if you want to make a simulation of Nature, you’d better make it quantum mechanical, and by golly it’s a wonderful problem, because it doesn’t look so easy”

R. Feyman, Simulating Physics with Computers, 1981
Quantum Information Science

Quantum Computation:
Use of well-controlled quantum systems for performing computations.
Exponential speed-ups over classical computing
E.g. Shor’s algorithm for factoring

Quantum Cryptography:
Use of well-controlled quantum systems for secret key distribution
Unconditional security based solely on the correctness of quantum mechanics
The Quest for Quantum Technology

- Cavity QED
- Optical Lattices
- Ion Trap
- Superconductors
- Quantum Dots
- Linear Optics
- NMR
Quantum Physics of Information

New ideas and tools developed in quantum information connect to physics more generally.
Quantum Physics of Information

New ideas and tools developed in quantum information connect to physics more generally

Example 1:
Can quantum computers be built from imperfect devices?

Quantum Error Correction

Condensed Matter (Topological Order)

High Energy, Gravity (Blackhole physics, Holography, AdS/CFT)
Quantum Physics of Information

New ideas and tools developed in quantum information connect to physics more generally

Example 2:
When does quantum offer an advantage for communication/cryptography?

Entanglement Theory

- Strongly Correlated Systems (i.e. Tensor Network states)
- High Energy (i.e. c-theorem, Ryu-Tagayanaki formula)
- Thermo/stat-mech (many-body localization, chaos, resource theories)
Quantum Physics of Information

New ideas and tools developed in quantum information connect to physics more generally.

Example 3:
When does quantum offers an advantage for computing?

Quantum Algorithms/Complexity

- Stat-Mech (i.e. spin glasses, stimulating annealing)
- Condensed matter (i.e. DFT, DMRG, ...)
- Quantum Field Theory (i.e. Jones polynomial, ...)

(above diagram with red arrows pointing to each category)
Entropy

\[ \log \text{number accessible states} \]

number of (q)bits of information
Entropy

log number accessible states

\[ k_B \ln \Omega \]

number of (q)bits of information

\[ - \sum_i p_i \log p_i \]
Entropy

You should call it entropy for two reasons: first because that is what the formula is in statistical mechanics, but second and more important, as nobody knows what entropy is, whenever you use the term you will always be at an advantage!
Quantum Entropy

Density matrix: $\rho$ acts on $\mathbb{C}^d$, $\rho \geq 0$, $\text{tr}(\rho) = 1$

Ex. Spin-1/2 particle: $\rho = \frac{1}{3} | \uparrow \rangle \langle \uparrow | + \frac{2}{3} | \rightarrow \rangle \langle \rightarrow |$

(a.k.a. qubit)
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Spectral decomposition: $\rho = \sum_i \lambda_i |\psi_i\rangle\langle\psi_i|$

{\lambda_i} form a probability distribution

Eigenvalues

Eigenvectors
Quantum Entropy

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(a.k.a. qubit)

Spectral decomposition: $\rho = \sum_i \lambda_i |\psi_i\rangle\langle\psi_i|$

$\{\lambda_i\}$ form a probability distribution

von Neumann entropy: $S(\rho) = -\text{tr}(\rho \log \rho) = -\sum_i \lambda_i \log \lambda_i$

Ex. $\rho = \frac{1}{2} |\uparrow\rangle\langle\uparrow| + \frac{1}{2} |\downarrow\rangle\langle\downarrow|$, $S(\rho) = 1$
Quantum Entropy

**Interpretation:** Entropy measures the effective number of qubits of the state

*(Shannon’s) Schumacher’s compression:*

For large $n$

\[
\rho \otimes \ldots \otimes \rho \approx \pi_n \\
\text{log rank}(\pi_n) \approx nS(\rho)
\]
Quantum Entropy

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(Shannon’s) Schumacher’s compression:
For large $n$

$$\rho \otimes \ldots \otimes \rho \approx \pi_n$$

$$\log \text{rank} (\pi_n) \approx nS(\rho)$$
Density matrix of two particles:

\[ \rho_{AB} \in \mathcal{D}(\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}) \]

Subadditivity:

\[ S(\rho_{AB}) \leq S(\rho_A) + S(\rho_B) \]
Entropy Inequalities

Density matrix of two particles:

$$\rho_{AB} \in D(\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B})$$

Subadditivity:

$$S(\rho_{AB}) \leq S(\rho_A) + S(\rho_B)$$

Density matrix three particles:

$$\rho_{ABC} \in D(\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B} \otimes \mathbb{C}^{d_C})$$

Strong subadditivity (SSA), (Lieb-Ruskai ‘73)

$$S(\rho_{AB}) + S(\rho_{BC}) \leq S(\rho_{ABC}) + S(\rho_B)$$

Strongest entropy inequality for 3 particles
Measuring Correlations

Mutual Information:
Measure of correlations of two quantum systems

For $\rho_{AB}$:

$$I(A : B)_\rho := S(\rho_A) + S(\rho_B) - S(\rho_{AB}) \geq 0$$

Ex 1. $|\uparrow, \uparrow\rangle$, $I(A:B) = 0$

Ex 2. $(|\uparrow, \uparrow\rangle + |\downarrow, \downarrow\rangle)/\sqrt{2}$, $I(A:B) = 2$

(Groisman et al ‘04) Measures how much information about $A$ must be discarded to destroy correlations with $B$
SSA as Data Processing Inequality

\[ S(AB) + S(BC) \geq S(ABC) + S(B) \]

\[ S(BC) - S(ABC) \geq S(B) - S(AB) \]

\[ S(A) + S(BC) - S(ABC) \geq S(A) + S(B) - S(AB) \]

\[ I(A : BC) \geq I(A : B) \]

Discarding information can only reduce correlations
Relative Entropy

...quantifies the distinguishability of two states

\[ S(\rho \| \sigma) = \text{tr}(\rho (\log \rho - \log \sigma)) \]

Positivity: \[ S(\rho \| \sigma) \geq 0 \]
Relative Entropy

...quantifies the distinguishability of two states

\[ S(\rho \| \sigma) = \text{tr}(\rho (\log \rho - \log \sigma)) \]

Positivity:

\[ S(\rho \| \sigma) \geq 0 \]

Data-Processing:

\[ S(\rho \| \sigma) \geq S(\Lambda(\rho) \| \Lambda(\sigma)) \]

\( \Lambda \): any physical operation (e.g. evolution Schrondiger equation, averaging out degrees of freedom)
Relative Entropy

...quantifies the distinguishability of two states

\[ S(\rho \parallel \sigma) = \text{tr}(\rho (\log \rho - \log \sigma)) \]

Positivity: \[ S(\rho \parallel \sigma) \geq 0 \]

Data-Processing: \[ S(\rho \parallel \sigma) \geq S(\Lambda(\rho) \parallel \Lambda(\sigma)) \]

Sub-additivity:

\[ S(A) + S(B) - S(AB) = S(\rho_{AB} \parallel \rho_A \otimes \rho_B) \geq 0 \]
Relative Entropy

...quantifies the distinguishability of two states

\[ S(\rho \| \sigma) = \text{tr}(\rho (\log \rho - \log \sigma)) \]

Positivity: \[ S(\rho \| \sigma) \geq 0 \]

Monotonicity: \[ S(\rho \| \sigma) \geq S(\Lambda(\rho) \| \Lambda(\sigma)) \]

Strong sub-additivity:

\[ S(AB) + S(BC) - S(ABC) - S(B) \]
\[ = S(\rho_{ABC} \| \rho_A \otimes \rho_{BC}) - S(\rho_{AB} \| \rho_A \otimes \rho_B) \geq 0 \]

In fact, SSA equivalent to monotonicity
Plan

Discuss a few applications of SSA:

1. Principle of minimum free energy
2. Other applications (coding, high energy, ...)

My own take on SSA applications:

3. Locality of Entanglement Spectrum
4. Quantum Algorithms for Semidefinite Programs
Plan

Discuss a few applications of SSA:

1. Principle of minimum free energy
2. Other applications (coding, high energy, …)

My own take on SSA applications:

3. Locality of Entanglement Spectrum
4. Quantum Algorithms for Semidefinite Programs
Minimum Free Energy Principle

“At constant temperature $T$, a system is at equilibrium when its free energy is at a minimum”

Free energy:  \[ F(\rho) = \text{tr}(H\rho) - TS(\rho) \]

Thermal equilibrium states:  \[ \rho_T = e^{-H/T}/Z(T) \]
Minimum Free Energy Principle

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Thermal equilibrium states: $\rho_T = e^{-H/T}/Z(T)$

Min $F$ Principle: $F(\rho) - F(\rho_T) = TS(\rho\|\rho_T) \geq 0$
Minimum Free Energy Principle

“At constant temperature $T$, a system is at equilibrium when its free energy is at a minimum”

Free energy: $F(\rho) = \text{tr}(H \rho) - TS(\rho)$

Thermal equilibrium states: $\rho_T = e^{-H/T}/Z(T)$

Min $F$ Principle: $F(\rho) - F(\rho_T) = TS(\rho \parallel \rho_T) \geq 0$

Why? $TS(\rho \parallel \rho_T) = T \left(-S(\rho) - \text{tr}(\rho \log \rho_T)\right)$

$= -TS(\rho) + \text{tr}(\rho H) + T \log Z(T)$

$= F(\rho) - F(\rho_T)$
Other Applications of SSA

Used to prove optimal rates for nearly every quantum information protocol.

- Channel capacities (classical, quantum, private)
- Distillable Entanglement
- Distillable Key Rate

(Casini, Huerta, ...) SSA + Lorentz Invariance:
- Entropic proof of the $c$-theorem
  (irreversibility of renormalization flow)
- Proof of Bekenstein’s and Bousso’s bound

(Ryu-Takayanagi, Headrick, ...) Test for holographic proposals of entropy
Entanglement

Entanglement in quantum information science is a resource (teleportation, quantum key distribution, ...)

Ex. EPR pair $|\phi^+\rangle = (|0, 0\rangle + |1, 1\rangle)/\sqrt{2}$
Entanglement in quantum information science is a **resource** (teleportation, quantum key distribution, ...)

Ex. EPR pair \( |\phi^+\rangle = (|0,0\rangle + |1,1\rangle)/\sqrt{2} \)

How to quantify it?

**Bipartite Pure State Entanglement**

Given \( |\psi\rangle_{AB} \), its entropy of entanglement is

\[
E(|\psi\rangle_{AB}) := S(\rho_A) = S(\rho_B)
\]

Reduced State: \( \rho_A := \text{tr}_B(|\psi\rangle\langle\psi|_{AB}) \)
Entanglement in Many-Body Systems

A quantum state $\psi$ of $n$ qubits is a vector in $\left(\mathbb{C}^2 \otimes n\right) \cong \mathbb{C}^{2^n}$

$$\left| \psi \right\rangle = \sum_{i_1, \ldots, i_n} c_{i_1, \ldots, i_n} \left| i_1, \ldots, i_n \right\rangle$$

For almost every state $\psi$, $S(X)_{\psi} \approx \text{vol}(X)$
Area Law

Quantum states on a lattice

**Def:** Area Law holds for $|\psi\rangle$ if for all R,

$$S(\text{tr}_{R^c}(|\psi\rangle\langle\psi|)) \leq O(|\partial R|)$$

- $\partial R$: boundary of R
- $|R|$: volume of R
- $|\partial R|$: volume of $\partial R$

Low energy states of matter expected to fulfil an area law

**Why interesting?** E.g. connected to the ability of simulating the system efficiently (tensor networks)
**Area Law**

Quantum states on a lattice

**Def:** Area Law holds for $\ket{\psi}$ if for all $R$,

$$S(\text{tr}_{R^c}(\ketbra{\psi}{\psi})) \leq O(|\partial R|)$$

- $\partial R$: boundary of $R$
- $|R|$: volume of $R$
- $|\partial R|$: volume of $\partial R$

Rigorous results? For gapped 1D models (Hastings ‘07, ...)

Open problem in higher dimensions
Intuition for Area Law

Reduced State on $R_i$: $\rho_i = \text{tr}_{R_i}(|\psi\rangle\langle\psi|)$

Assumption 1: $\rho_i$ all the same

Assumption 2: specific heat capacity behavior close to zero:

$$c(T) \leq T^{-\nu} e^{-\Delta/T}$$

Expected in gapped (massive) systems

Implication: $S(\rho_i) \leq O(l \log(l))$
Reduced State on $R_i$: $\rho_i = \text{tr}_{R_i}(|\psi\rangle\langle\psi|)$

Define: $\pi = \rho R_1 \otimes \ldots \otimes \rho R_{L^2/l^2}$

We have:

$$\text{tr}(\pi H) \leq \langle\psi|H|\psi\rangle + O\left(\frac{L^2}{l^2}l\right)$$
Intuition for Area Law

Reduced State on $R_i$:  
$$  \rho_i = \text{tr}_{R_i}(|\psi\rangle\langle\psi|) $$

Define:  
$$  \pi = \rho_{R_1} \otimes \ldots \otimes \rho_{R_{L^2/l^2}} $$

We have:
$$  \text{tr}(\pi H) \leq \langle \psi | H | \psi \rangle + O \left( \frac{L^2}{l^2} l \right) $$

Let temperature $T$ be s.t.  
$$  \text{tr}(H \rho_T) = \text{tr}(H \pi) $$

Mean energy:  
$$  u(T) := \frac{\text{tr}(H \rho_T)}{L^2} \lesssim \frac{1}{l} $$
Intuition for Area Law

Reduced State on $R_i$: \[ \rho_i = \text{tr}_{R_i} (|\psi\rangle \langle \psi|) \]

Define: \[ \pi = \rho_{R_1} \otimes \cdots \otimes \rho_{R_{L^2/l^2}} \]

Mean-energy: \[ u(T) := \text{tr}(H \rho_T)/L^2 \lesssim 1/l \]

Principle min free energy:
\[ F(\pi) \geq F(\rho_T) \implies S(\rho_j) = \frac{l^2}{L^2} S(\pi) \leq \frac{l^2}{L^2} S(\rho_T) \]

Need to show that \[ s(T) := \frac{1}{L^2} S(\rho_T) \lesssim O(l \log(l)) \]
Intuition for Area Law

Reduced State on $R_i$: $\rho_i = \text{tr}_{R_i}(|\psi\rangle\langle\psi|)$

Define: $\pi = \rho_{R_1} \otimes \ldots \otimes \rho_{R_{L^2/l^2}}$

Mean-energy: $u(T) := \text{tr}(H \rho_T)/L^2 \lesssim 1/l$

Need to show that $s(T) := \frac{1}{L^2} S(\rho_T) \lesssim O(l \log(l))$

Follows from $s(T) = s(0) + \int_0^T \frac{c(T')}{T'} dT'$

$u(T) = u(0) + \int_0^T c(T') dT'$

and assumption $c(T) \leq T^{-\nu} e^{-\Delta/T}$
"Uniform" Area Law

For every region $X$,

$$S(X) = a|\partial X| - \gamma + \exp(-c|\partial|/\xi)$$

Topological entanglement entropy (TEE)
(Kitaev, Preskill '05, Levin, Wen '05)

correlation length

Expected to hold in models with a finite correlation length $\xi$.
E.g. lowest energy state of gapped local Hamiltonians
For every region \( X \),

\[
S(X) = a|\partial X| - \gamma + \exp(-c|\partial|/\xi)
\]

Topological entanglement entropy (TEE)

(Kitaev, Preskill ‘05, Levin, Wen ‘05)

Expected to hold in models with a finite correlation length \( \xi \).

TEE \( \gamma \) accounts for long-range quantum entanglement in the system (i.e. entanglement that cannot be created by short local dynamics)

\( \gamma \neq 0 \): Fractional Quantum Hall, Spin liquids, Toric code, ...

What are the consequences of an area law?
Measuring Conditional Correlations

Conditional Mutual Information (CMI):
Measure of correlations of two quantum systems relative to a third. For $\rho_{ABC}$:

$$I(A : C|B)_{\rho} := S(AB) + S(BC) - S(ABC) - S(B) \geq 0$$

For a probability distribution $p_{XYZ}$

$$I(X : Y|Z) = \mathbb{E}_{z' \sim p(z)} I(X : Y)_{p(x,y|z=z')}$$

No simple relation for quantum in general
CMI vs TEE

Area law assumption: For every region \( X \),

\[
S(X) = a|\partial X| - \gamma + \exp(-c|\partial|/\xi)
\]

Topological entanglement entropy

For every ABC with trivial topology:

\[
I(A : C|B) \leq \exp(-cl)
\]
Area law assumption: For every region $X$,

$$S(X) = a|\partial X| - \gamma + \exp(-c|\partial|/\xi)$$

Topological entanglement entropy

Correlation length

$$I(A : C|B)$$

$$= S(AB) + S(BC) - S(ABC) - S(B)$$

$$= a(|\partial A| + |\partial B| + |\partial C| - |\partial A| - |\partial B| - |\partial C| - |\partial B|) + 2\gamma + \exp(-cl)$$

$$= 2\gamma + \exp(-cl)$$

$$\gamma \approx I(A : C|B)/2$$
Entanglement Spectrum

\[ \lambda(\rho_X) \text{: eigenvalues of reduced density matrix on } X \]

\[ \rho_R = \text{tr}_{R^c}(|\psi\rangle\langle\psi|) \]

\(\lambda(\rho_X)\) is called entanglement spectrum

Area law is an statement about \(- \sum_{i} \lambda_i \log \lambda_i\)

Are there more information in the distribution of the \(\lambda_i\)?
Entanglement Spectrum

$|\psi\rangle_{RR^c}$

$\lambda(\rho_X)$: eigenvalues of reduced density matrix on $X$

$\rho_R = \text{tr}_{R^c}(|\psi\rangle\langle\psi|)$

$\lambda(\rho_X)$ is called entanglement spectrum

Write $\rho_R = e^H$, with $H = \log \rho_R$

Entanglement spectrum is equivalent (up to taking an exponential) to the spectrum of $H$
Entanglement Spectrum

\[ |\psi\rangle_{RR^c} \]

\[ \lambda(\rho_X) : \text{eigenvalues of reduced density matrix on } X \]

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\( \lambda(\rho_X) \) is called entanglement spectrum

Write \( \rho_R = e^H \), with \( H = \log \rho_R \)

Entanglement spectrum is equivalent (up to taking an exponential) to the spectrum of \( H \)

What can we say about \( H \)?
Area law for \( \rho_R \) suggests \( H \) should act on the boundary of \( R \)
Entanglement Spectrum

\[ \lambda(\rho_X) \] : eigenvalues of reduced density matrix on X

\[ \rho_R = \text{tr}_{R^c}(|\psi\rangle\langle\psi|) \]

(Haldane, Li ’08, ....)

Fractional Quantum Hall State: entanglement spectrum related to chiral CFT on boundary
**Entanglement Spectrum**

\[ \lambda(\rho_X) : \text{eigenvalues of reduced density matrix on } X \]

\[ \rho_R = \text{tr}_{R^c} (|\psi\rangle\langle\psi|) \]

(Haldane, Li ’08, ....)

Fractional Quantum Hall State: entanglement spectrum related to chiral CFT on boundary

(Cirac, Poiblanc, Schuch, Verstraete ’11, ....)

Numerical studies lattice systems with tensor network states:

For *topologically trivial* systems (AKLT, Heisenberg model, ...): entanglement spectrum matches the energies of a *local Hamiltonian* on boundary

For *topological* systems (Toric code): needs *non-local* Hamiltonian
Entanglement Spectrum

\[ |\psi\rangle_{RR^c} \]

\[ \lambda(\rho_X) : \text{eigenvalues of reduced density matrix on X} \]

\[ \rho_R = \text{tr}_{R^c}(|\psi\rangle\langle\psi|) \]

Question:
How general is this connection?
Can we prove it from first principles?

We show **SSA implies it**, only assuming an area law holds.
Suppose $|\psi\rangle$ satisfies the area law assumption with $\gamma = 0$. Then

$$\lambda(\rho_X)^{\otimes 2} \approx \lambda(\epsilon \sum_k H_{B_k, B_{k+1}})$$

$$\lambda(\rho_X)^{\otimes 2} = \lambda(\rho_X \otimes \rho_X)$$

$$= \lambda(\rho_X \otimes \rho_X^*)$$

$$l \approx \xi \log L$$
Entanglement Spectrum -> Boundary State

\[ I(X : X') \]
\[ = I(X : X'|B) \approx 0 \]

\[ \rho_{XX'} \approx \rho_X \otimes \rho_{X'} \]
Entanglement Spectrum -> Boundary State

\[
I(X : X') = I(X : X'|B) \approx 0
\]

\[
\rho_{XX'} \approx \rho_X \otimes \rho_{X'}
\]

\[
\lambda(\rho_{XX'}) = \lambda(\rho_B)
\]
Entanglement Spectrum -> Boundary State

\[ I(X : X') = I(X : X' | B) \approx 0 \]

\[ \rho_{XX'} \approx \rho_X \otimes \rho_{X'} \]

\[ \lambda(\rho_{XX'}) = \lambda(\rho_B) \]

\[ \lambda(\rho_X) \otimes \lambda(\rho_{X'}) \approx \lambda(\rho_B) \]
Entanglement Spectrum -> Boundary State

\[ I(X : X') = I(X : X' | B) \approx 0 \]
\[ \rho_{XX'} \approx \rho_X \otimes \rho_{X'} \]
\[ \lambda(\rho_{XX'}) = \lambda(\rho_B) \]
\[ \lambda(\rho_X) \otimes \lambda(\rho_{X'}) \approx \lambda(\rho_B) \]

Suffices to show that if \( \gamma = 0 \),
\[ \rho_B \approx e^{\sum_k H_{B_k, B_{k+1}} / Z} \]
Reduced State is Thermal

Want to show that if $\gamma = 0$,

$$\rho_B \approx e^{\sum_k H_{B_k, B_{k+1}} / Z}$$

By area law:

$$I(B_{i-1} : B_{i+1} \ldots B_{2k} | B_i) \leq e^{-l/\xi}$$

The idea is to show this implies the state is approximately thermal
Idea of the proof

Let $\sigma_{B_1 \ldots B_{2k}}$ be the maximum entropy state s.t.

$$\sigma_{B_i, B_{i+1}} = \rho_{B_i, B_{i+1}}$$

$$2k = L/l$$
Idea of the proof

Let \( \sigma_{B_1 \ldots B_{2k}} \) be the maximum entropy state s.t.

\[
\sigma_{B_i, B_{i+1}} = \rho_{B_i, B_{i+1}}
\]

Fact 1 (Jaynes’ Principle): \( \sigma = \exp \left( \sum_{k} H_{B_k, B_{k+1}} \right) \)
Idea of the proof

Let $\sigma_{B_1...B_{2k}}$ be the maximum entropy state s.t.

$$\sigma_{B_i,B_{i+1}} = \rho_{B_i,B_{i+1}}$$

**Fact 1 (Jaynes’ Principle):**

$$\sigma = \exp\left(\sum_k H_{B_k,B_{k+1}}\right)$$

**Fact 2**

$$\min_{H \in 2l\text{-local}} S(\rho \| e^H) \leq -S(\rho) - \text{tr}(\rho \log(\sigma))$$

$$= S(\sigma) - S(\rho)$$

Let’s show it’s small
Idea of the proof

\[ S(B_1 \ldots B_{2k})_\sigma \leq S(B_1 B_2)_\sigma - S(B_2)_\sigma + S(B_2 \ldots B_{2k})_\sigma \]

\[ 2k = \frac{L}{l} \]
Idea of the proof

\[ S(B_1 \ldots B_{2k})_\sigma \]
\[ \leq S(B_1 B_2)_\sigma - S(B_2)_\sigma + S(B_2 \ldots B_{2k})_\sigma \]
\[ \leq S(B_1 B_2)_\sigma - S(B_2)_\sigma + S(B_2 B_3)_\sigma - S(B_3)_\sigma + S(B_3 \ldots B_{2k})_\sigma \]

\[ 2k = L/l \]
Idea of the proof

\[ 2k = \frac{L}{l} \]

\[
S(B_1 \ldots B_{2k})_\sigma \\
\leq S(B_1 B_2)_\sigma - S(B_2)_\sigma + S(B_2 \ldots B_{2k})_\sigma \\
\leq S(B_1 B_2)_\sigma - S(B_2)_\sigma + S(B_2 B_3)_\sigma - S(B_3)_\sigma + S(B_3 \ldots B_{2k})_\sigma \\
\leq \sum_{i} S(B_i B_{i+1})_\sigma - S(B_{i+1})_\sigma
\]
Idea of the proof

\[ S(B_1 \ldots B_{2k})_\sigma \]
\[ \leq S(B_1 B_2)_\sigma - S(B_2)_\sigma + S(B_2 \ldots B_{2k})_\sigma \]
\[ \leq S(B_1 B_2)_\sigma - S(B_2)_\sigma + S(B_2 B_3)_\sigma - S(B_3)_\sigma + S(B_3 \ldots B_{2k})_\sigma \]
\[ \leq \sum_i S(B_i B_{i+1})_\sigma - S(B_{i+1})_\sigma \]
\[ = \sum_i S(B_i B_{i+1})_\rho - S(B_{i+1})_\rho \]

\[ 2k = L/l \]
Idea of the proof

\[ I(B_{i-1}: B_{i+1} \ldots B_{2k} | B_i) \leq e^{-\frac{l}{\xi}} \]

\[
S(B_1 \ldots B_{2k})_\sigma \\
\leq S(B_1 B_2)_\sigma - S(B_2)_\sigma + S(B_2 \ldots B_{2k})_\sigma \\
\leq S(B_1 B_2)_\sigma - S(B_2)_\sigma + S(B_2 B_3)_\sigma - S(B_3)_\sigma + S(B_3 \ldots B_{2k})_\sigma \\
\leq \sum_i S(B_i B_{i+1})_\sigma - S(B_{i+1})_\sigma \\
= \sum_i S(B_i B_{i+1})_\rho - S(B_{i+1})_\rho \\
\leq S(B_1 \ldots B_{2k})_\rho + e^{-\frac{l}{\xi}} L/l \]
Suppose $|\psi\rangle$ satisfies the area law assumption. Then

$$2\gamma \approx I(A : C | B)$$

$$\approx \min_{H_{AB}, H_{BC}} S(\rho_{ABC} \parallel \exp(H_{AB} + H_{BC})/Z)$$

Proof uses recent strengthening of SSA (Fawzi-Renner ‘14)

$$I(A : C | B) \geq \min_{\Lambda : B \to BC} \log F(\rho_{ABC}, I_A \otimes \Lambda(\rho_{AB}))$$

F: quantum fidelity
Quantum Algorithms

Once we build a quantum computer, what will we do with it?

Exponential speed-ups:
Simulate quantum physics, factor big numbers (Shor’s algorithm), …,

Polynomial Speed-ups:
Searching (Grover’s algorithm: $N^{1/2}$ vs $O(N)$), …

Heuristics:
Quantum annealing (adiabatic algorithm), machine learning, …
Quantum Algorithms

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Exponential speed-ups:
Simulate quantum physics, factor big numbers (Shor’s algorithm), ...,

Polynomial Speed-ups:
Searching (Grover’s algorithm: $N^{1/2}$ vs $O(N)$), ...,

Heuristics:
Quantum annealing (adiabatic algorithm), machine learning, ...

We’ll use data processing inequality to find a new quantum algorithm in these two classes
Semidefinite Programming

... is an important class of convex optimization problems

\[
\max \, \text{tr}(C X) \\
\forall j \in [m], \quad \text{tr}(A_j X) \leq b_j \\
X \geq 0.
\]

Input: \( n \times n \) matrices \( r \)-sparse \( C, A_1, ..., A_m \) and numbers \( b_1, ..., b_m \)

Output: \( X \)
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Some Applications: operations research (location problems, scheduling, ...), bioengineering (flux balance analysis, ...), approximating NP-hard problems (max-cut, ...), field theory (conformal bootstrapping), ...
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Algorithms

- Interior points: \(O((m^2nr + mn^2)\log(1/\varepsilon))\)
- Multilicative Weights: \(O((mnr/\varepsilon^2))\)
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Algorithms

- Interior points: \(O((m^2nr + mn^2)\log(1/\varepsilon))\)
- Multiplicative Weights: \(O((mnr/\varepsilon^2))\)

Lower bound: No faster than \(\Omega(nm)\), for constant \(\varepsilon\) and \(r\)
Quantum Algorithm for Semidefinite Programming

\[ \max \operatorname{tr}(CX) \]

\[ \forall j \in [m], \quad \operatorname{tr}(A_j X) \leq b_j \]

\[ X \geq 0. \]

Worst Case Running time: \( O((nm)^{1/2}r/\varepsilon^2) \)

Quantum Lower Bound: \( \Omega((nm)^{1/2}) \), for constant \( \varepsilon \) and \( r \)

Gives square-root speed-ups both in \( n \) and \( m \)

with Krysta Svore (MSR)
Quantum Algorithm for Semidefinite Programming

with Krysta Svore (MSR)

\[ \max \text{tr}(C X) \]

\[ \forall j \in [m], \quad \text{tr}(A_j X) \leq b_j \]

\[ X \geq 0. \]

The algorithm boils down to preparing on a quantum computer quantum Gibbs states of the form:

\[ \exp \left( \frac{\sum \lambda_j A_j + \nu C}{Z} \right) \]

Grover-type speed-up on preparation gives \( n^{1/2} \) dependence

It’s also an interesting heuristic for larger speed-ups
Quantum Algorithm for Semidefinite Programming

with Krysta Svore (MSR)

\[
\max \operatorname{tr}(CX)
\]

\[
\forall j \in [m], \quad \operatorname{tr}(A_jX) \leq b_j \leq b_j \\
X \succeq 0.
\]

The quantum algorithm is based on a classical algorithm for Semidefinite programming, based on the matrix multiplicative weights method (Arora-Kale ‘07).

The method is derived using monotonicity quantum relative entropy (Warmuth, Kuzmin ‘06)
Conclusion

• Quantum entropy inequalities are powerful. SSA is a fundamental constraint on quantum states of three particles. Are there more applications?

• Synergy of quantum information and other areas of physics is exciting. Can we deepen the connections?

Thanks!