

Faithful Squashed Entanglement

with applications to separability testing and
quantum Merlin-Arthur games

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Mutual Information vs Conditional Mutual Information

Mutual Information: Measures the correlations of **A** and **B** in ρ_{AB}

$$I(A:B)_\rho := S(A)_\rho + S(B)_\rho - S(AB)_\rho$$

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Approximate version? Pinsker's inequality:

$$I(A:B) \geq \frac{1}{2 \ln 2} \left\| \rho_{AB} - \rho_A \otimes \rho_B \right\|_1^2$$

Remark: dimension-independent! Useful in many application in QIT (e.g. decoupling, QKD, ...)

Mutual Information vs Conditional Mutual Information

Conditional Mutual Information: Measures the correlations of **A** and **B** relative to **E** in ρ_{ABE}

$$I(A:B | E)_\rho := S(AE)_\rho + S(BE)_\rho - S(ABE)_\rho - S(E)_\rho$$

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$I(A:B|E)_\rho = 0$ **iff** ρ_{ABE} is a “**Quantum Markov Chain State**”

(Hayden, Jozsa, Petz, Winter '04)

E.g.
$$\rho_{ABE} = \sum_k p_k \rho_k^A \otimes \rho_k^B \otimes |k\rangle^E \langle k|$$

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Approximate version??? ...

Outline

- $I(A:B | E) \approx 0$ characterization
- Applications:
 - Squashed Entanglement**
 - de Finetti-type bounds**
 - Algorithm for Separability**
 - A new characterization of QMA**
- Proof

No-Go For Approximate Version

A naïve guess for approximate version (à la Pinsker):

$$I(A : B | E) \stackrel{?}{\geq} \Omega \left(\min_{\sigma = \sum_k p_k \sigma_A^k \otimes \sigma_B^k \otimes |k\rangle_E \langle k|} \left\| \rho_{ABE} - \sigma_{ABE} \right\|_1^2 \right) \geq \Omega \left(\min_k \left\| \rho_{AB} - \sigma_{AB} \right\|_1^2 \right)$$

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||

$O(|A|^{-1})$

It fails badly!

||

$\Omega(1)$

E.g. Antisymmetric Werner state (Christandl, Schuch, Winter '08)

Main Result

Thm: (B., Christandl, Yard '10)

$$I(A : B | E) \geq \Omega \left(\min_{\sigma \in SEP} \left\| \rho_{AB} - \sigma_{AB} \right\|^2 \right)$$

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Pointed out yesterday
by David Reeb

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The **Euclidean (Frobenius) norm**:

$$\|X\|_2 = \text{tr}(X^T X)^{1/2}$$

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(Euclidean norm or ~~(one-way)~~ LOCC norm)

The trace norm:

$$\|X\|_1 = \frac{1}{2} + \frac{1}{2} \max_{0 \leq A \leq I} |\text{tr}(AX)|$$

$\|\rho - \sigma\|_1$: optimal bias

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The LOCC norm:

$$\|X\|_{\text{LOCC}} = \frac{1}{2} + \frac{1}{2} \max_{0 \leq A \leq I} |\text{tr}(AX)| : \{A, I-A\} \text{ in LOCC}$$

$\|\rho - \sigma\|_{\text{LOCC}}$: optimal bias by LOCC

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The one-way LOCC norm:

$$\|X\|_{\text{LOCC}(1)} = \frac{1}{2} + \frac{1}{2} \max_{0 \leq A \leq I} |\text{tr}(AX)| : \{A, I-A\} \text{ in one-way LOCC}$$

$\|\rho - \sigma\|_{\text{LOCC}}$: optimal bias by one-way LOCC

The Power of LOCC

Thm: (B., Christandl, Yard '10)

$$I(A : B | E) \geq \Omega \left(\min_{\sigma \in SEP} \left\| \rho_{AB} - \sigma_{AB} \right\|^2 \right)$$

(Euclidean norm or ~~(one-way)~~ LOCC norm)

(Matthews, Wehner, Winter '09) For X in $A \otimes B$

$$\|X\|_1 \geq \|X\|_{LOCC} \geq \|X\|_{LOCC \rightarrow} \geq \Omega(\|X\|_2) \geq \Omega\left(\left(|A||B|\right)^{-1/2} \|X\|_1\right)$$

Interesting one, uses a covariant
random local measurement

Squashed Entanglement

(Christandl, Winter '04) **Squashed entanglement:**

$$E_{\text{sq}}(\rho_{AB}) = \inf_{\pi} \left\{ \frac{1}{2} I(A:B|E)_{\pi} : \text{tr}_E(\pi_{ABE}) = \rho_{AB} \right\}$$

Open question: **Is it faithful?**

i.e. Is $E_{\text{sq}}(\rho_{AB}) > 0$ for every entangled ρ_{AB} ?

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Corollary:
$$E_{sq}(\rho_{AB}) \geq \Omega\left(\min_{\sigma \in SEP} \|\rho - \sigma\|_{LOCC}^2\right)$$

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Corollary $E_{sq}(\rho_{AB}) \geq \Omega\left(\min_{\sigma \in SEP} \|\rho - \sigma\|_{LOCC}^2\right)$

Proof:

From $I(A:B|E) \geq \Omega\left(\min_{\sigma \in SEP} \|\rho_{AB} - \sigma_{AB}\|_{LOCC(1)}\right)$

Follows: $E_{sq}(\rho_{AB}) \geq \Omega\left(\min_{\sigma \in SEP} \|\rho - \sigma\|_{LOCC(1)}^2\right)$

General two-way LOCC: **monotonicity of squashed entanglement under LOCC**

Entanglement Zoo

Measure	E_{sq}	E_D	K_D	E_C	E_F	E_R	E_R^∞	E_N
normalisation	y	y	y	y	y	y	y	y
faithfulness	y	n	?	y	y	y	y	n
LOCC monotonicity	y	y	y	y	y	y	y	y
asymptotic continuity	y	?	?	?	y	y	y	n
convexity	y	?	?	?	y	y	y	n
strong superadditivity	y	y	y	?	n	n	?	?
subadditivity	y	?	?	y	y	y	y	y
monogamy	y	?	?	n	n	n	n	?

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subadditivity	y	?	?	y	y	y	y	y
monogamy	y	?	?	n	n	n	n	?

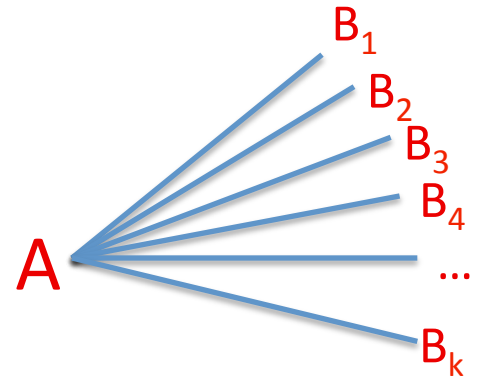


Entanglement Monogamy

Classical correlations are shareable:

$$\sigma_{AB_1, \dots, B_k} = \sum_j p_j \sigma_{A,j} \otimes \sigma_{B,j}^{\otimes k}$$

Def. ρ_{AB} is *k-extendible* if there is $\rho_{AB_1 \dots B_k}$
s.t for all j in $[k]$ $\text{tr}_{\setminus B_j}(\rho_{AB_1 \dots B_k}) = \rho_{AB}$



Separable states are k-extendible for every k.

Entanglement Monogamy

Quantum correlations are non-shareable:

ρ_{AB} entangled iff ρ_{AB} not k -extendible for some k

- Follows from: **Quantum de Finetti Theorem** (Stormer '69, Hudson & Moody '76, Raggio & Werner '89)

- E.g.** - Any pure entangled state is not 2-extendible
- The $d \times d$ antisymmetric state is not d -extendible

Entanglement Monogamy

Quantitative version: For any k -extendible ρ_{AB} ,

$$\min_{\sigma \in SEP} \|\rho - \sigma\|_1 \leq O\left(\frac{|B|^2}{k}\right)$$

- Follows from: **finite quantum de Finetti Theorem** (Christandl, König, Mitchson, Renner '05)

Entanglement Monogamy

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- Follows from: **finite quantum de Finetti Theorem** (Christandl, König, Mitchson, Renner '05)

Close to optimal:

there is a state ρ_{AB} s.t. $\min_{\sigma \in SEP} \|\rho - \sigma\|_1 \geq \Omega\left(\frac{|B|}{k}\right)$
(guess which? 😊)

For **other norms** ($\|\cdot\|_2, \|\cdot\|_{LOCC}, \dots$) no better bound known.

Exponentially Improved de Finetti type bound

Corollary For any k -extendible ρ_{AB} , with $\|\cdot\|$ equals $\|\cdot\|_2$ or $\|\cdot\|_{\text{LOCC}}$

$$\min_{\sigma \in \text{SEP}} \|\rho - \sigma\| \leq O\left(\frac{\log |A|}{k}\right)^{\frac{1}{2}}$$

Bound proportional to the (square root) of the number of qubits: exponential improvement over previous bound

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Proof: E_{sq} satisfies monogamy relation (Koashi, Winter '05)

$$E_{sq}(\rho_{A:B\bar{B}}) \geq E_{sq}(\rho_{A:B}) + E_{sq}(\rho_{A\bar{B}})$$

For ρ_{AB} k -extendible:

$$\log|A| \geq E_{sq}(\rho_{A:B_1\dots B_k}) \geq kE_{sq}(\rho_{AB}) \geq kO\left(\min_{\sigma \in \text{SEP}} \|\rho - \sigma\|^2\right)$$

Exponentially Improved de Finetti type bound

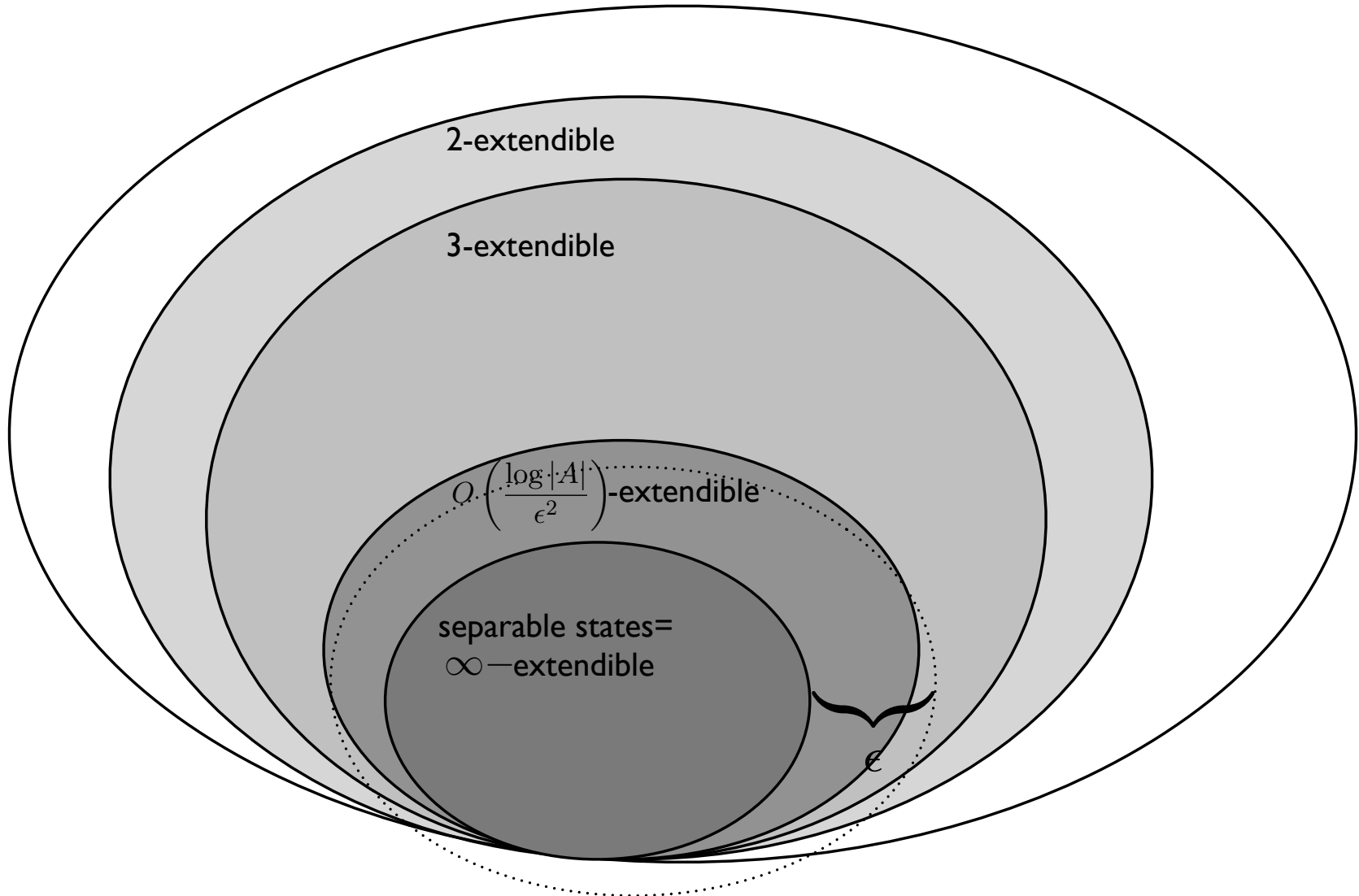
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(Close-to-Optimal) There is k -extendible state ρ_{AB} s.t.

$$\min_{\sigma \in SEP} \|\rho - \sigma\|_{LOCC} \geq \Omega\left(\frac{\log|A|}{k}\right)$$

Exponentially Improved de Finetti type bound



The separability problem

When is ρ_{AB} entangled?

- Decide if ρ_{AB} is separable or ε -away from separable

Beautiful theory behind it (PPT, entanglement witnesses, symmetric extensions, etc)

Horribly expensive algorithms

State-of-the-art: $2^{O(|A| \log(1/\varepsilon))}$ time complexity

(Doherty, Parrilo, Spedalieri '04)

The separability problem

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- Decide if ρ_{AB} is separable or ε -away from separable

Hardness results:

(Gurvits '02) NP-hard with $\varepsilon=1/\exp(d)$ ($d:= (|A| |B|)^{1/2}$)

(Gharibian '08, Beigi '08) NP-hard with $\varepsilon=1/\text{poly}(d)$

(Beigi&Shor '08) Favorite separability tests fail

(Harrow&Montanaro '10) No $\exp(O(d^{2-\delta}))$ time algorithm for membership in any convex set within $\varepsilon=\Omega(1)$ trace distance to SEP, unless ETH fails

ETH (Exponential Time Hypothesis): SAT cannot be solved in $2^{o(n)}$ time
(Impagliazzo&Paruti '99)

Quasi-polynomial Algorithm

Corollary There is a $\exp(O(\varepsilon^{-2} \log |A| \log |B|))$ time algorithm for deciding separability (in $\|\cdot\|_2$ or $\|\cdot\|_{\text{LOCC}}$)

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The idea (Doherty, Parrilo, Spedalieri '04)

Search for a $k=O(\log |A|/\varepsilon^2)$ extension of ρ_{AB} by SDP

$$\exists \pi_{AB_1, \dots, B_k} \geq 0 : \pi_{AB_j} = \rho_{AB} \quad \forall j \in [k]$$

Complexity SDP of size

$$|A|^2 |B|^{2k} = \exp(O(\varepsilon^{-2} \log |A| \log |B|))$$

Quasi-polynomial Algorithm

Corollary There is a $\exp(O(\varepsilon^{-2} \log |A| \log |B|))$ time algorithm for deciding separability (in $\|\cdot\|_2$ or $\|\cdot\|_{\text{LOCC}}$)

NP-hardness for $\varepsilon = 1/\text{poly}(d)$ is shown using $\|\cdot\|_2$

From corollary: the problem in $\|\cdot\|_2$ **cannot be NP-hard** for $\varepsilon = 1/\text{polylog}(d)$, unless ETH fails

Best Separable State Problem

BSS(ϵ) Problem: Given X , approximate to additive error ϵ $\max_{|a\rangle, |b\rangle} \langle a, b | X | a, b \rangle$

Corollary There is a $\exp(O(\epsilon^{-2} \log |A| \log |B| (\|X\|_2)^2))$ time algorithm for **BSS(ϵ)**

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The idea Optimize over $k=O(\log |A| \epsilon^{-2} (\|X\|_2)^2)$ extension of ρ_{AB} by SDP

$$\min_{\pi} \text{tr}(\pi X) : \pi_{AB_1, \dots, B_k} \geq 0, \quad \pi_{AB_j} = \rho_{AB} \quad \forall j \in [k]$$

Best Separable State Problem

BSS(ϵ) Problem: Given X , approximate $\max_{|a\rangle, |b\rangle} \langle a, b | X | a, b \rangle$
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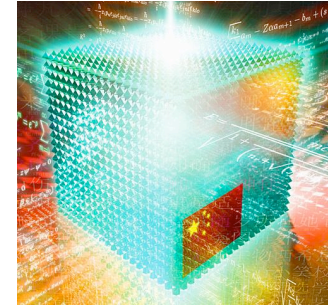
Corollary There is a $\exp(O(\epsilon^{-2} \log |A| \log |B| (\|X\|_2)^2))$ time algorithm for **BSS(ϵ)**

(Harrow and Montanaro '10): **BSS(ϵ)** for $\epsilon = \Omega(1)$ and $\|X\|_\infty \leq 1$ cannot be solved in $\exp(O(\log^{1-\nu} |A| \log^{1-\mu} |B|))$ time for any $\nu + \mu > 0$ unless ETH fails

QMA



$|\Psi\rangle$



**Quantum
Computer**

A language L is in QMA if for every x in L :

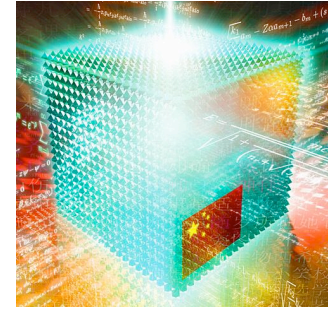
QMA:

- YES instance: Merlin can convince Arthur with probability $> 2/3$

QMA



$|\Psi\rangle$



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A language L is in QMA if for every x in L :

QMA:

- YES instance: Merlin can convince Arthur with probability $> 2/3$
- NO instance: Merlin cannot convince Arthur with probability $> 1/3$

QMA

- Quantum analogue of NP (or MA)
- Local Hamiltonian Problem, ...

Is QMA a robust complexity class?

(Aharonov, Regev '03) superverifiers doesn't help

(Marriott, Watrous '05) Exponential amplification with fixed proof size

(Beigi, Shor, Watrous '09) logarithmic size interaction doesn't help

New Characterization QMA

Corollary QMA doesn't change allowing $k = O(1)$ different proofs if the verifier can only apply LOCC measurements in the k proofs

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New Characterization QMA

Corollary QMA doesn't change allowing $k = O(1)$ different proofs if the verifier can only apply **LOCC measurements** in the k proofs

Def $\text{QMA}_{m,s,c}(k)$: analogue of QMA with k proofs, proof size m , soundness s and completeness c .

Def $\text{LOCCQMA}_{m,s,c}(k)$: analogue of QMA with k proofs, proof size m , soundness s , completeness c and **LOCC verification** procedure along the k proofs.

New Characterization QMA

Corollary $\text{QMA} = \text{LOCCQMA}(k), \quad k = O(1)$

$$\text{LOCCQMA}_{m,s,c}(2) = \text{QMA}_{O(m^2\varepsilon^{-2}),s+\varepsilon,c}$$

Contrast: $\text{QMA}_{m,s,c}(2)$ not in $\text{QMA}_{O(m^{2-\delta}\varepsilon^{-2}),s+\varepsilon,c}$
for $\varepsilon = O(1)$ and $\delta > 0$ unless **Quantum ETH*** fails

Follows from Harrow and Montanaro '10 (based on Aarason et al '08)

* **Quantum ETH:** SAT cannot be solved in $2^{o(n)}$ quantum time

New Characterization QMA

Corollary $\text{QMA} = \text{LOCCQMA}(k), \quad k = O(1)$

$$\text{LOCCQMA}_{m,s,c}(2) = \text{QMA}_{O(m^2\varepsilon^{-2}),s+\varepsilon,c}$$

Idea to simulate $\text{LOCCQMA}_{m,s,c}(2)$ in QMA:

- Arthur asks for proof ρ on $AB_1B_2\dots B_k$ with $k = m\varepsilon^{-2}$
- He **symmetrizes** the B systems and apply the original verification procedure to AB_1

Correctness

de Finetti bound implies: $\min_{\sigma \in \text{SEP}} \left\| \rho_{AB_1} - \sigma \right\|_{\text{LOCC}} \leq \sqrt{\frac{m}{k}} = \varepsilon$

Proof

Relative Entropy of Entanglement

The proof is largely based on the properties of a *different* entanglement measure:

Def Relative Entropy of Entanglement (Vedral, Plenio '99)

$$E_R^\infty(\rho_{AB}) := \lim_{n \rightarrow \infty} \frac{E_R(\rho_{AB}^{\otimes n})}{n} \quad E_R(\rho_{AB}) := \min_{\sigma \in SEP} S(\rho \parallel \sigma)$$

$$S(\rho \parallel \sigma) := \text{tr}(\rho(\log \rho - \log \sigma))$$

Entanglement Hypothesis Testing

Given (many copies) of ρ_{AB} , what's the optimal probability of distinguishing it from a separable state?

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Def Rate Function: $D(\rho_{AB})$ is maximum number s.t. there exists $\{M_n, I-M_n\}$, $0 < M_n < I$,

$$\min_{\sigma \in SEP} \text{tr}(M_n \sigma) \leq 2^{-nr}, \quad \text{tr}(M \rho_{AB}^{\otimes n}) \geq \Omega(1)$$

$D_{LOCC}(\rho_{AB})$: defined analogously, but now $\{M, I-M\}$ must be LOCC

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$D_{LOCC}(\rho_{AB})$: defined analogously, but now $\{M, I-M\}$ must be LOCC

(B., Plenio '08) $D(\rho_{AB}) = E_R^\infty(\rho_{AB})$

Obs: Equivalent to reversibility of entanglement under non-entangling operations

Proof in 1 Line

$$I(A : B | E)_{\rho_{ABE}} \stackrel{(i)}{\geq} E_R^\infty(\rho_{A:BE}) - E_R^\infty(\rho_{A:E}) \stackrel{(ii)}{\geq} D_{LOCC}(\rho_{A:B}) \stackrel{(iii)}{\geq} \Omega\left(\min_{\sigma \in SEP} \|\rho_{A:B} - \sigma\|_{LOCC(1)}^2\right)$$

Proof in 1 Line

$$I(A : B | E)_{\rho_{ABE}} \stackrel{(i)}{\geq} E_R^\infty(\rho_{A:BE}) - E_R^\infty(\rho_{A:E}) \stackrel{(ii)}{\geq} D_{LOCC}(\rho_{A:B}) \stackrel{(iii)}{\geq} \Omega\left(\min_{\sigma \in SEP} \|\rho_{A:B} - \sigma\|_{LOCC(1)}^2\right)$$

Relative entropy of Entanglement plays a double role:

- (i) **Quantum Shannon Theory:** State redistribution Protocol
(Devetak and Yard '07)
- (ii) **Large Deviation Theory:** Entanglement Hypothesis Testing
(B. and Plenio '08)
- (iii) **Entanglement Theory:** Faithfulness bounds

First Inequality

$$I(A : B | E)_{\rho_{ABE}} \stackrel{(i)}{\geq} E_R^\infty(\rho_{A:BE}) - E_R^\infty(\rho_{A:E})$$

Non-lockability: $E_R(\rho_{A:BE}) \leq E_R(\rho_{A:E}) + 2 \log |B|$
(Horodecki³ and Oppenheim '04)

State Redistribution: How much does it cost to redistribute a quantum system? $\frac{1}{2} I(A:B | E)$

$$A \mid BE \mid F \longrightarrow A \mid E \mid BF \quad |\psi\rangle_{A:BE:F}^{\otimes n} \longrightarrow |\psi\rangle_{A:E:BF}^{\otimes n}$$

(i) Apply non-lockability to $\rho_{A:BE}^{\otimes n}$ and use state redistribution to trace out B at a rate of $\frac{1}{2} I(A:B | E)$ qubits per copy

Second Inequality

$$E_R^\infty(\rho_{A:BE}) - E_R^\infty(\rho_{A:E}) \stackrel{(ii)}{\geq} D_{\text{LOCC}(1)}(\rho_{A:B})$$

Equivalent to: $D(\rho_{A:BE}) \geq D(\rho_{A:E}) + D_{\text{LOCC}(1)}(\rho_{A:B})$

Monogamy relation for entanglement hypothesis testing

Idea Use optimal measurements for ρ_{AE} and ρ_{AB} achieving $D(\rho_{AE})$ and $D_{\text{LOCC}(1)}(\rho_{AB})$, resp., to **construct a measurement** for $\rho_{A:BE}$ achieving $D(\rho_{A:BE})$

Third Inequality

$$D_{LOCC(1)}(\rho_{A:B}) \stackrel{(iii)}{\geq} \Omega\left(\min_{\sigma \in SEP} \|\rho_{A:B} - \sigma\|_{LOCC(1)}^2\right)$$

Pinsker type inequality for entanglement hypothesis testing

Idea **minimax theorem** + **martingale like property** of the set of separable states

Open Question

- Can we prove a lower bound on $I(A:B|E)$ in terms of distance to “markov quantum chain states”?
- Can we close the LOCC norm vs. trace norm gap in the results (hardness vs. algorithm, LOCCQMA(k) vs QMA(k))?
- Are there more applications of the bound on the convergence of the SDP relaxation?
- Are there more application of the main inequality?

Thanks!