Electromagetism QFT

Now recall for a massless axion (spin zero particle) we introduced a (~~classical field~~) real field

$$a(\vec{x}) = \int \frac{d^3k}{(2\pi)^{3/2}} E_k \left(e^{i\vec{k}\cdot\vec{x}} b(\vec{k}) + e^{-i\vec{k}\cdot\vec{x}} b^\dagger(\vec{k}) \right)$$

& in Heisenberg picture

$$a_H(\vec{x},t) = \int \frac{d^3k}{(2\pi)^{3/2}} E_k \Big(e^{i(\vec{k}\cdot\vec{x} - \frac{E_k t}{\hbar})} b(\vec{k})$$
$$+ e^{-i(\vec{k}\cdot\vec{x} - \frac{E_k t}{\hbar})} b^\dagger(\vec{k}) \Big)$$

$$E_k = \hbar\omega_k \qquad \omega_k = c|\vec{k}| = ck$$

We wanto do same for photons or electromagnetic field It will be a real field we introduce operators that destroy & create photons with wave vector $\vec{k}$ & polarization $\lambda = 1, 2$. $a(\vec{k},\lambda)$, $a^\dagger(\vec{k},\lambda)$

$$[a(\vec{k},\lambda), a^\dagger(\vec{k}',\lambda')] = \delta^{\lambda\lambda'}\delta^3(\vec{k}-\vec{k}')$$

For electromagnetism we work in gauge $\phi = 0$. with no sources in this gauge

$$\vec{B} = \vec{\nabla}\times\vec{A}, \quad \vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t}$$

Maxwells eqs

$$\vec{\nabla}\cdot\vec{E} = 0, \quad \vec{\nabla}\times\vec{B} - \frac{1}{c}\frac{\partial \vec{E}}{\partial t} = 0, \quad \vec{\nabla}\times\vec{E} + \frac{1}{c}\frac{\partial \vec{B}}{\partial t} = 0, \quad \vec{\nabla}\cdot\vec{B} = 0$$

automatic

automatic

We used gauge invariance $\phi \to \phi + \partial_t \Omega$ to set $\phi = 0$. We can still make transformations $\Omega = \Omega(\vec{x})$ that are independent of time with ϕ staying zero. Then we can set

$$\vec{\nabla} \cdot \vec{A} = 0$$

but only at one time. But

$$\vec{\nabla} \cdot \vec{E} = 0 \implies \frac{\partial}{\partial t}\left(\vec{\nabla} \cdot \vec{A}\right) = 0$$

so for all times $\vec{\nabla} \cdot \vec{A} = 0$. So our non interacting photons are described by a vector field $\vec{A}(\vec{x}, t)$ that satisfies

$$\vec{\nabla} \cdot \vec{A} = 0$$

Now we want to quantize. $\vec{A}(\vec{x})$ in Schrodinger picture is observable + real

$$\vec{A}(\vec{x}) = \int \frac{d^3k}{(2\pi)^{3/2}} N(k) \left[a(\vec{k}, \lambda) \vec{\epsilon}(\vec{k}, \lambda) e^{i\vec{k}\cdot\vec{x}} + a^\dagger(\vec{k}, \lambda) \vec{\epsilon}(\vec{k}, \lambda)^* e^{-i\vec{k}\cdot\vec{x}} \right]$$

Now $\vec{\epsilon}$ is polarization vector of photon which is part of its one particle wf. Recall for spin 1/2 particle with mom $\vec{p} = \hbar\vec{k}$

$$\psi = \frac{e^{i\vec{k}\cdot\vec{x}}}{(2\pi\hbar)^{3/2}} \chi_s$$

$\chi_s \longrightarrow$ spin part of wf + $\vec{\epsilon}$ is analog of this

$\vec{\nabla} \cdot \vec{A} = 0 \Rightarrow \vec{k} \cdot \bar{\epsilon}(\vec{k}, \lambda) = 0$

So only 2 polarization. Choose ~~$\epsilon \cdot \epsilon =$~~

$\vec{\epsilon}(\vec{k}, \lambda) \cdot \vec{\epsilon}(\vec{k}, \lambda') = \delta_{\lambda\lambda'}$

So for $\vec{k} = (0, 0, k)$ can take

$\bar{\epsilon}(\vec{k}, 1) = (1, 0, 0)$, $\bar{\epsilon}(\vec{k}, 2) = (0, 1, 0)$

We still need to know $N(k)$. We want

$$H = \sum_\lambda \int d^3k \; \hbar\omega_k \; a^\dagger(\vec{k}, \lambda) a(\vec{k}, \lambda) \quad *$$

($\hbar\omega_k \to E_k$)

+ (possible irrelevant constant)

But we also have classical expression for energy

$$\underset{\substack{\text{classical}\\\text{formula}}}{E} = \frac{1}{8\pi} \int d^3x \left(|\vec{E}|^2 + |\vec{B}|^2 \right)$$

OK go to Heisenberg picture

$$\vec{A}_H(\bar{x}, t) = \sum_\lambda \int \frac{d^3k}{(2\pi)^{3/2}} N(k) \left[a(\vec{k}, \lambda) \bar{\epsilon}(\vec{k}, \lambda) e^{i(\vec{k}\cdot\bar{x} - E_k t/\hbar)} + a^\dagger(\vec{k}, \lambda) \vec{\epsilon}^{\,*}(\vec{k}, \lambda) e^{-i(\vec{k}\cdot\bar{x} - E_k t/\hbar)} \right]$$

Then ~~take~~ use $\bar{E} = -\frac{1}{c}\frac{\partial \bar{A}}{\partial t}$, $\bar{B} = \bar{\nabla} \times \bar{A}$ to get E, B.

~~At~~ this stage can ~~plug~~ go back to Schrodinger picture by setting $t = 0$ + plug into $E_{classical}$ formula + find ~~the~~ $N(k)$ that gives Hamiltonian in (*). Will put on problem set but find

$$N(k) = \sqrt{\frac{2\pi c^2 \hbar}{\omega_k}}$$

So

$$\vec{A}_H(\vec{x},t) = \sum_\lambda \int \frac{d^3k}{(2\pi)^{3/2}} \sqrt{\frac{2\pi c^2\hbar}{\omega_k}} \Big[a(\vec{k},\lambda)\,\vec{\epsilon}(\vec{k},\lambda)\, e^{i(\vec{k}\cdot\vec{x} - E_k t/\hbar)} + a^\dagger(\vec{k},\lambda)\,\vec{\epsilon}^{\,*}(\vec{k},\lambda)\, e^{-i(\vec{k}\cdot\vec{x} - E_k t/\hbar)} \Big]$$

and in the usual Shrodinger picture

$$\vec{A}(\vec{x}) = \sum_\lambda \int \frac{d^3k}{(2\pi)^{3/2}} \sqrt{\frac{2\pi c^2\hbar}{\omega_k}} \Big[a(\vec{k},\lambda)\,\vec{\epsilon}(\vec{k},\lambda)\, e^{i\vec{k}\cdot\vec{x}} + a^\dagger(\vec{k},\lambda)\,\vec{\epsilon}^{\,*}(\vec{k},\lambda)\, e^{-i\vec{k}\cdot\vec{x}} \Big]$$

Spontaneous Atomic Decay

Consider decay of excited Hydrogen Atom

$$|\overset{n}{2}, \overset{\ell}{\ell}, \overset{m}{m}\rangle \rightarrow |1,0,0\rangle + \text{photon}$$

Use Fermis Golden Rule. Initial state

$$|i^{(0)}\rangle = |2,\ell,m\rangle \underbrace{|0\rangle}_{\text{no photon}}$$

Final state

$$|f^{(0)}\rangle = |\overset{n}{1}, \overset{\ell}{0}, \overset{m}{0}\rangle\, |\vec{k},\lambda\rangle$$

$$H^{(1)} = \frac{e}{mc}\vec{A}(\vec{x})\cdot\vec{P}$$

Rate

$$R_{fi} = \frac{2\pi}{\hbar}\left|\langle f^{(0)}|H^{(1)}|i^{(0)}\rangle\right|^2 \delta\left(E_f^{(0)} - E_i^{(0)}\right)$$

Now we plug in field $\vec{A}(x)$

$$\langle f^{(0)}|H^{(1)}|i^{(0)}\rangle = \sum_{\lambda'} \sqrt{\frac{\hbar c^2}{(2\pi)^2}} \int \frac{d^3k'}{\sqrt{\omega_{k'}}}$$

$$\langle \vec{k},\lambda|a^\dagger(\vec{k}',\lambda')|0\rangle\, \epsilon^*(\vec{k}',\lambda')$$

$$\langle 1,0,0|\frac{e}{mc}e^{i\vec{k}'\cdot\hat{x}}\cdot\hat{P}|2,l,m\rangle$$

Now $\hbar|\vec{k}'| = \frac{\Delta E^{\text{atomic}}}{c}$ small so as

below use dipole approx $e^{i\vec{k}'\cdot\hat{x}} \approx 1$. Next

$\cancel{\langle f^{(0)}|}$ $\langle \vec{k},\lambda|a^\dagger(\vec{k}',\lambda')|0\rangle = \delta^3(\vec{k}'-\vec{k})\delta^{\lambda\lambda'}$

& use this to do $\sum_{\lambda'}\int d^3k'$ & get

$$\langle f^{(0)}|H^{(1)}|i^{(0)}\rangle = \sqrt{\frac{\hbar c^2}{(2\pi)^2\omega_k}}\;\frac{e}{mc}\,\vec{\epsilon}^*(\vec{k},\lambda)$$

$$\cdot\langle 1,0,0|\hat{P}|2,l,m\rangle$$

In delta function for rate $E_f^{(0)} = E_{n=1}^{(\text{hydrogen})} + \hbar\omega_k$

$$E_i^{(0)} = E_{n=2}^{(\text{hydrogen})}$$

Get finite rate, no delta funct from summing over final photon state

$$\sum_\lambda \int d^3k \quad R_{fi}$$

Now recall $\qquad H^{(0)} = \frac{\vec{P}^2}{2m} + V(r)$

$$\langle 1\,0\,0 |\ P^j \,| 2\,l,m \rangle$$

$$= \left(\frac{m}{i\hbar}\right) \langle 1,0,0 |\ [x^j, H^{(0)}]\ | 2,l,m \rangle$$

$$= \left(\frac{m}{i\hbar}\right) \left(E_2^{(hydrogen)} - E_1^{(hydrogen)}\right) \langle 1,0,0 | x^j | 2,l,m \rangle$$

energy conservation

$$= \left(\frac{m}{i\cancel{\hbar}}\right) \cancel{\hbar}\omega_k \langle 1,0,0 | x^j | 2,l,m \rangle$$

$$R_{fi} = \left(\frac{2\pi}{\cancel{\hbar}}\right) \frac{\cancel{\hbar} e^2}{(2\pi)^2 \omega_k} \left(\frac{e^2}{\cancel{m^2} \cancel{c^2}}\right) \cancel{m^2} \omega_k^2$$

$$\left| \langle 1,0,0 | \vec{\epsilon}^{\,*}(\vec{k},\lambda) \cdot \vec{r} | 2,l,m \rangle \right|^2 \delta\left(E_2^{(hydrogen)} - E_1^{(hydrogen)} - \hbar\omega_k\right)$$

$$= \frac{\omega_k e^2}{2\pi} \left| \langle 1,0,0 | \vec{\epsilon}(\vec{k},\lambda) \cdot \vec{r} | 2,l,m \rangle \right|^2 \delta\left(-\frac{R_y}{4} + R_y - \hbar\omega_k\right)$$

For Homework you will do $\sum_\lambda \int d^3k$ to get final expression for the rate.