Quantum Approximate Markov Chains and the Locality of Entanglement Spectrum

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based on joint work with

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Markov Chain

Classical: X, Y, Z with distribution p(x, y, z)

i) *X-Y-Z* Markov if *X* and *Z* are independent conditioned on *Y*

ii) *X-Y-Z* Markov if there is a channel $\Lambda : Y \to YZ$ s.t. $\Lambda(p_{XY}) = p_{XYZ}$

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$$\rho_{ABC} = \bigoplus_k p_k \rho_{AB_{L,k}} \otimes \rho_{B_{R,k}C}$$

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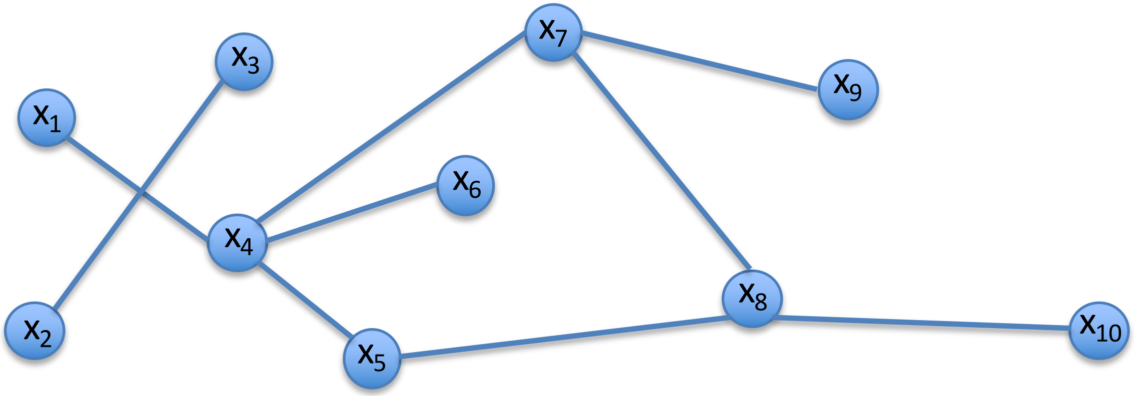
ii) ρ_{ABC} Markov if there is channel $\Lambda : B \to BC$ s.t. $\Lambda(\rho_{AB}) = \rho_{ABC}$

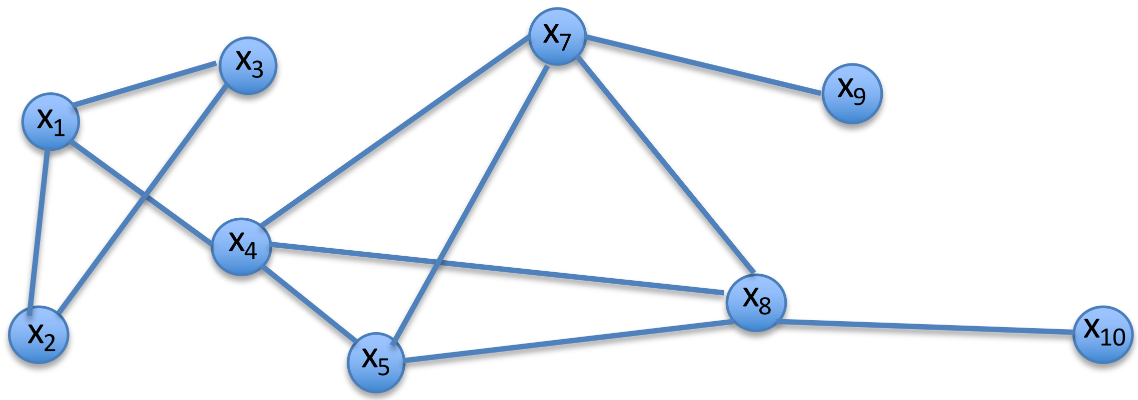
iii) ρ_{ABC} Markov if $\rho_{ABC} = e^{H_{AB}+H_{BC}}, \quad [H_{AB}, H_{BC}] = 0$

Markov Networks



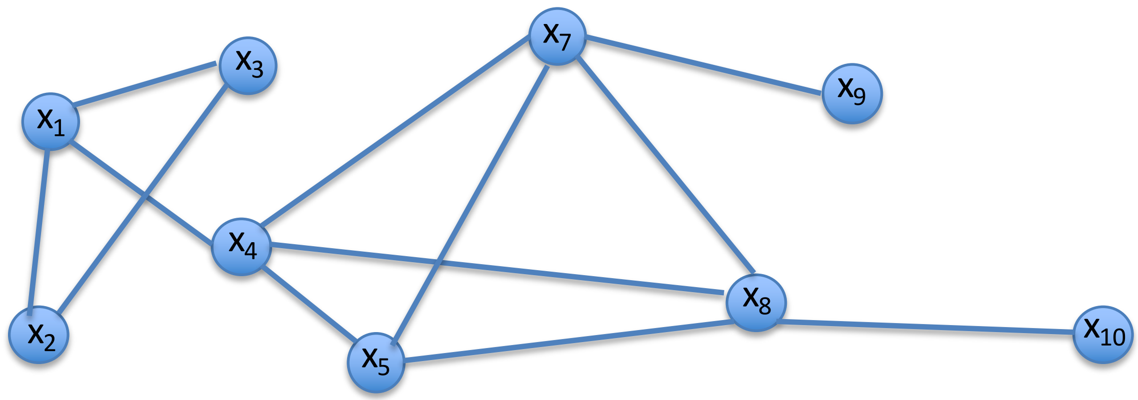
We say $X_1, \ldots, X_n$ on a graph G form a Markov Network if X_i *is* indendent of all other X's conditioned on its neighbors

Hammersley-Clifford Theorem



Markov networks ⟺ Gibbs state local classical Hamiltonian

Hammersley-Clifford Theorem

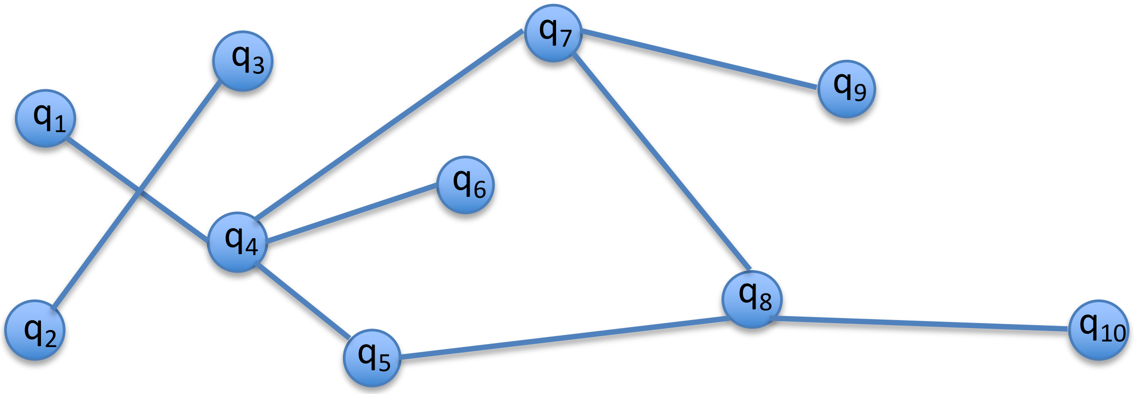


(Hammersley-Clifford '71) Let G = (V, E) be a graph and P(V) be a positive probability distribution over random variables located at the vertices of G. The pair (P(V), G) is a Markov Network if, and only if, the probability P can be expressed as $P(V) = e^{H(V)}/Z$ where

$$H(V) = \sum_{Q \in \mathcal{C}} h_Q(Q)$$

is a sum of real functions $h_Q(Q)$ of the random variables in cliques Q.

Quantum Hammersley-Clifford Theorem



(Leifer, Poulin '08, Brown, Poulin '12) Analogous result holds replacing classical Hamiltonians by *commuting* quantum Hamiltonians

(obs: quantum version more fragile; only works for graphs with no 3-cliques)

Only Gibbs states of commuting Hamiltonians appear. Is there a fully quantum formulation?

Approximate Conditional Independence

X-*Y*-*Z* approximate Markov if *X* and *Z* are approximately independent conditioned on *Y*

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Conditional Mutual Information:

$$I(X:Y|Z) = \mathop{\mathbb{E}}_{z' \sim p(z)} I(X:Y)_{p(x,y|z=z')}$$

Fact: $I(X:Y|Z)_p = \min_{q \in \mathrm{MC}} S(p\|q)$

Def: *X*-*Y*-*Z* ε-Markov if I(X:Y|Z) ≤ ε

Quantum Approximate Conditional Independence

ρ_{ABC} quantum approximate Markov if *A* and *C* are approximately independent conditioned on *B*

Quantum Conditional Mutual Information:

$$I(A:C|B) = S(AB) + S(BC) - S(ABC) - S(B)$$

Note: $I(X:Y|Z)_p \neq \min_{q \in \text{MC}} S(p\|q)$

But: (Fawzi, Renner '14)

$$I(A:C|B) \approx 0 \implies I_A \otimes \Lambda^{B \to BC}(\rho_{BC}) \approx \rho_{ABC}$$

Approximate recovery

Quantum Approximate Conditional Independence

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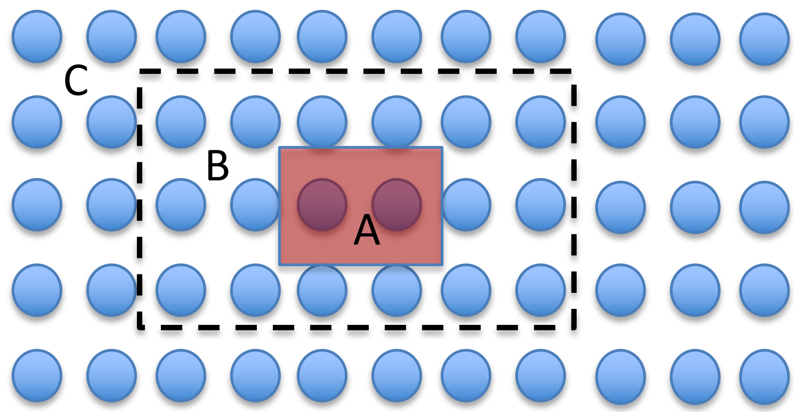
Quantum Conditional Mutual Information:

$$I(A:C|B) = S(AB) + S(BC) - S(ABC) - S(B)$$

Def: ρ_{ABC} quantum ε-Markov if $I(A:C|B) \leq \varepsilon$

Q. Approximate Markov States

ρ



ρ quantum approximate Markov if for every A, B, C

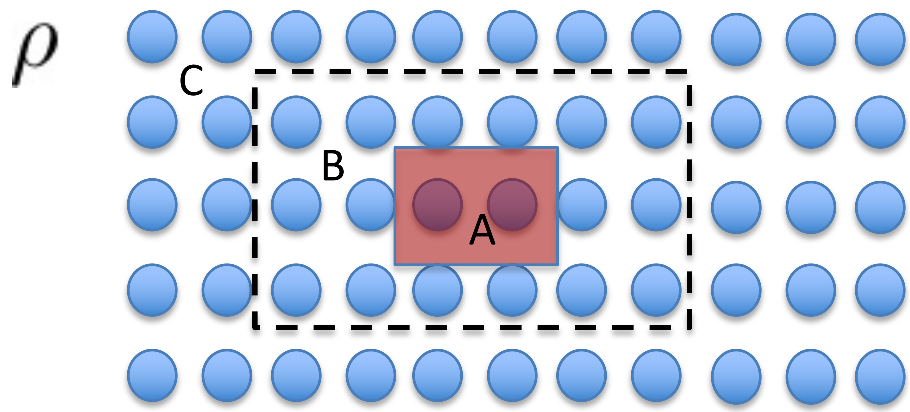
$I(A:C|B) \to 0$ when $\mathrm{dist}(A,C) \to \infty$

Conjecture

Quantum Approximate Markov ⟺ Gibbs state local Hamiltonian

$$\rho = e^{\sum_k H_k}$$

Strengthening of Area Law



Conjecture

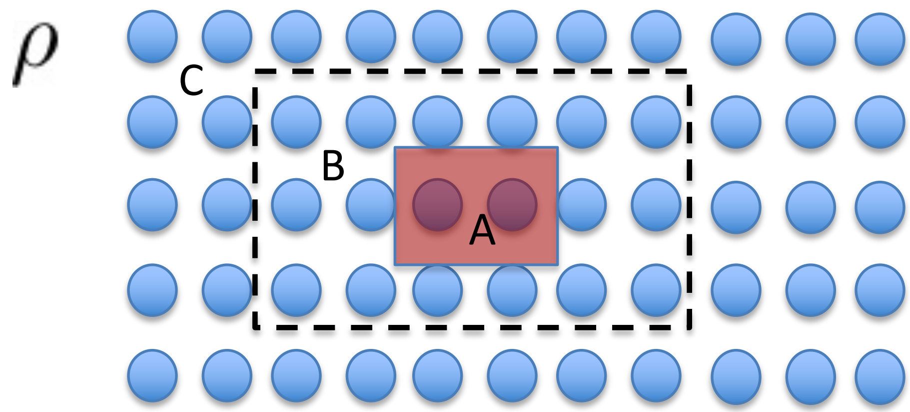
Quantum Approximate Markov ⟸ Gibbs state local Hamiltonian

(Wolf, Verstraete, Hastings, Cirac '07) $I(A:BC)_{\rho_T} \leq \frac{c}{T}|\partial A|$

Gibbs state @ temperature *T*: $\rho_T := e^{-H/T}/Z$

$$H = \sum_k H_k, \quad \|H_k\| \leq 1$$

Strengthening of Area Law



Conjecture

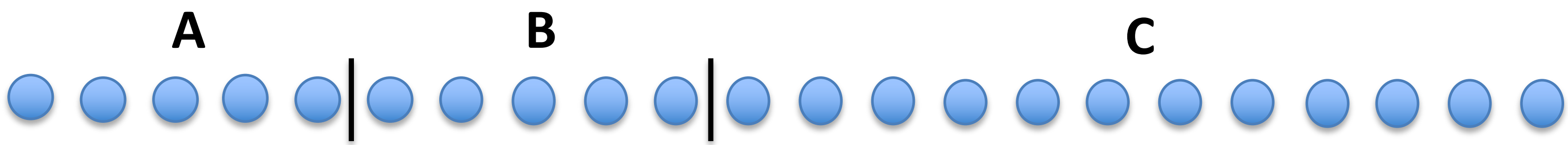
Quantum Approximate Markov ⟸ Gibbs state local Hamiltonian

From conjecture:

$$I(A:BC) = I(A:B) + I(A:C|B) \approx I(A:B)$$

Gives rate of saturation of area law

Approximate Quantum Markov Chains are Thermal



thm

1. Let *H* be a local Hamiltonian on *n* qubits. Then

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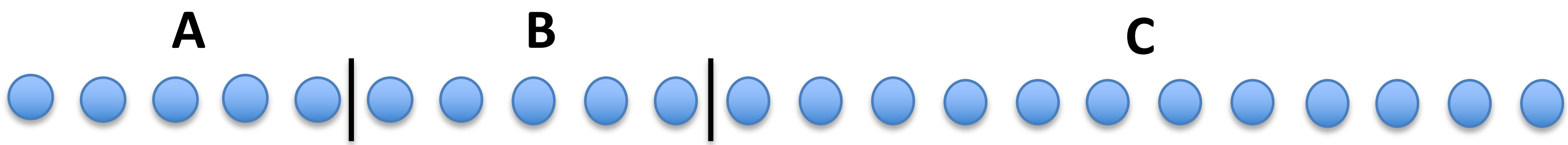

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$$\min_{H \in \mathcal{H}_{2m}} S\left(\rho || e^{H}\right) \leq \varepsilon \frac{n}{m}$$

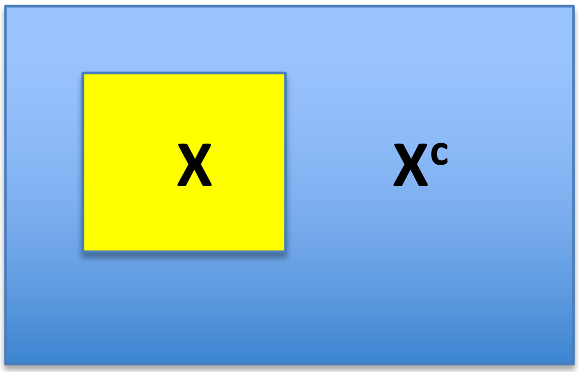
$$\mathcal{H}_{2m} := \{H : H = \sum_k H_{k,k+1},\ \forall k \ \text{supp}(H_{k,k+1}) \leq 2m\}$$

Rest of the Talk

- Application of theorem to Entanglement Spectrum
- Idea of the Proof

“Uniform” Area Law

$|\psi\rangle_{XX^c}$



For every region *X*,

$$S(X) = a|\partial X| - \gamma + \exp(-c|\partial|/\xi)$$

Topological entanglement entropy (TEE) (Kitaev, Preskill ‘05, Levin, Wen ‘05)

correlation length

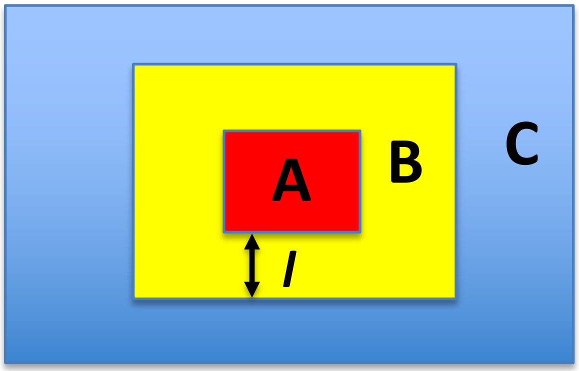
Expected to hold in models with a finite correlation length ξ.

TEE γ accounts for long-range quantum entanglement in the system (i.e. entanglement that cannot be created by short local dynamics)

$\gamma \neq 0$: Fractional Quantum Hall, Spin liquids, Toric code, …

What are the consequences of an area law?

Consequence of Area Law: Approximate Conditional Independece



Area law assumption: For every region *X*,

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Topological entanglement entropy

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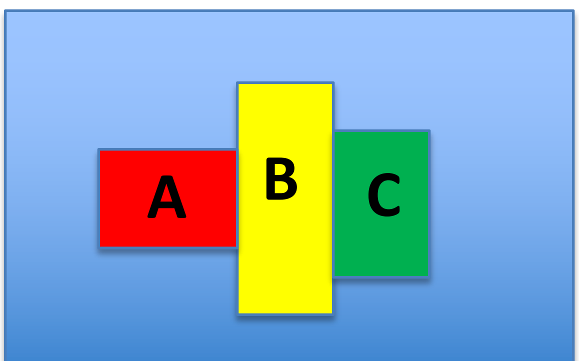
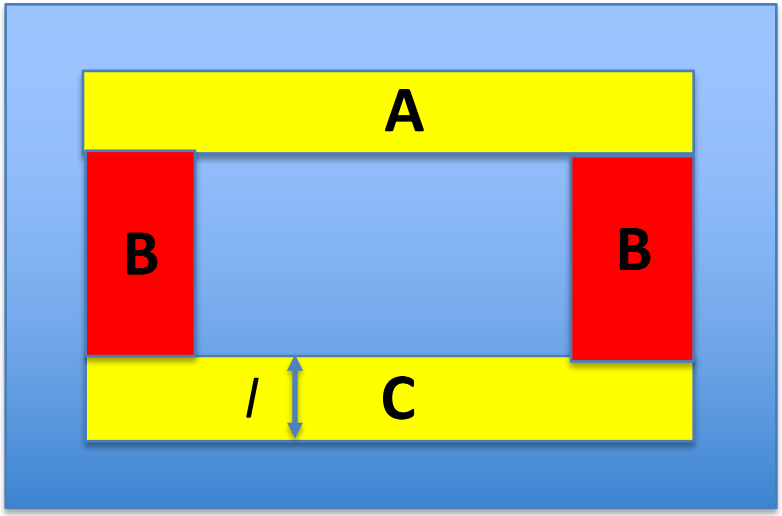


For every ABC with trivial topology:

$$I(A:C|B) \leq \exp(-cl)$$

Topological Entanglement Entropy

(Kitaev, Preskill '05, Levin, Wen '05)



Area law assumption: For every region *X*,

$$S(X) = a|\partial X| - \gamma + \exp(-c|\partial|/\xi)$$

Topological entanglement entropy

correlation length

Conditional Mutual Information:

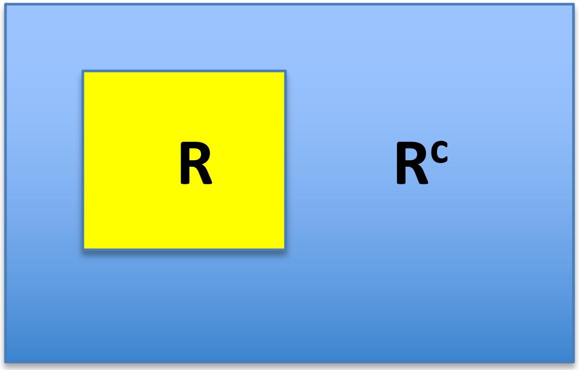
$$I(A:C|B) := S(AB) + S(BC) - S(ABC) - S(B)$$

Assuming area law holds:

$$I(A:C|B) = 2\gamma + \exp(-c'l)$$

Entanglement Spectrum

$|\psi\rangle_{XX^c}$



$\lambda(\rho_X)$: eigenvalues of reduced density matrix on X

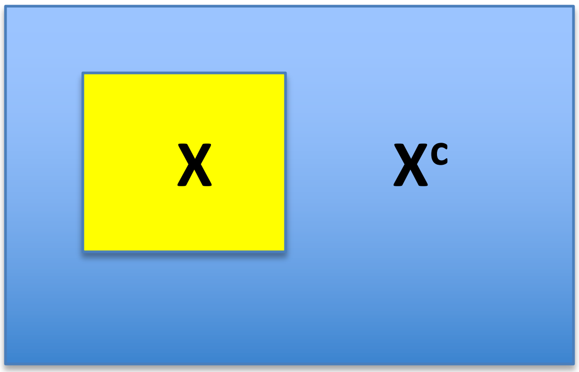
$\lambda(\rho_X)$ is called entanglement spectrum

Area law is an statement about $-\sum_i \lambda_i \log \lambda_i$

Are there more information in the *distribution* of the λ_i?

Entanglement Spectrum

$|\psi\rangle_{XX^c}$



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Also known as Schmidt eigenvalues of the state $|\psi\rangle_{XX^c}$

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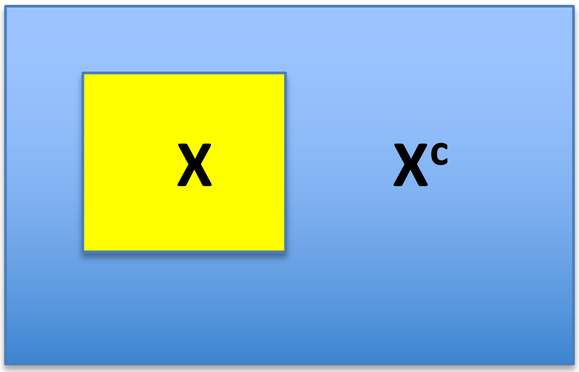

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(Cirac, Poiblanc, Schuch, Verstraete '11, ….)

Numerical studies with PEPS

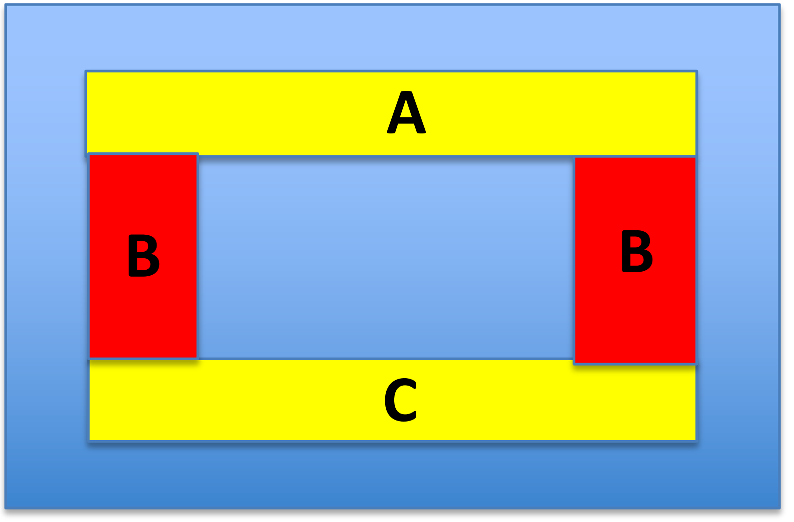
For *topologically trivial* systems (AKLT, Heisenberg models):
entanglement spectrum matches the energies of a *local Hamiltonian* on boundary

How generic are these findings? Can we give a proof?

Result 1: Boundary State

claim 1 Suppose $|\psi\rangle$ satisfies the area law assumption. Then

$$\begin{aligned} 2\gamma &\approx I(A:C|B) \\ &\approx \min_{H_{AB},H_{BC}} S(\rho_{ABC}\|\exp(H_{AB}+H_{BC})/Z) \end{aligned}$$



Result 1: Boundary State

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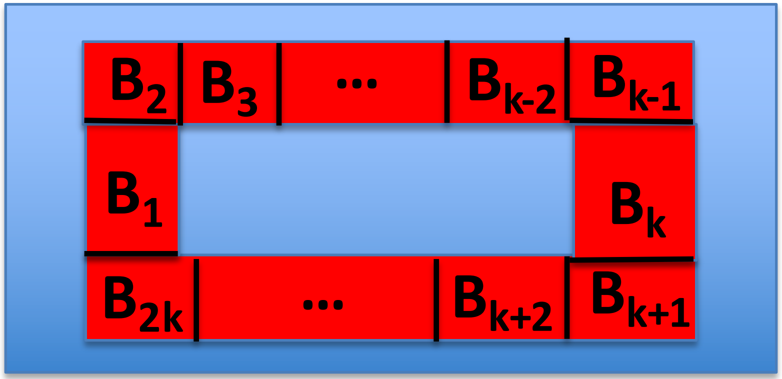
$$\rho_{B_1 \ldots B_l} \approx \exp(H)/Z$$



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In general:

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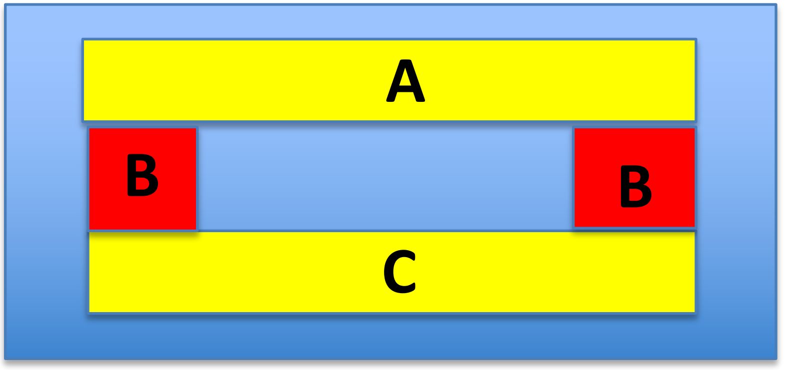

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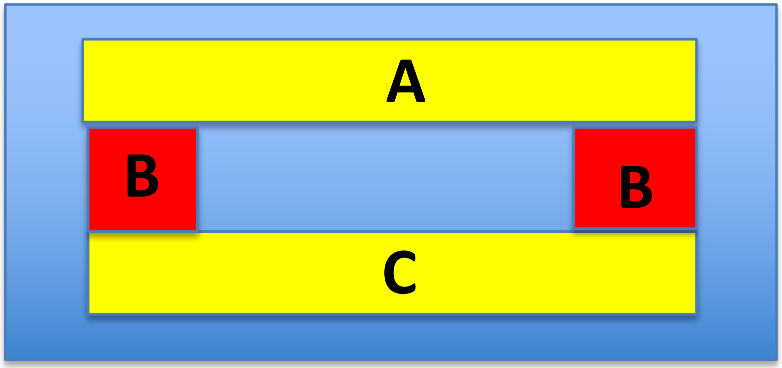
Zero-correlation case proved by (Kato, Furrer, Murao '15)



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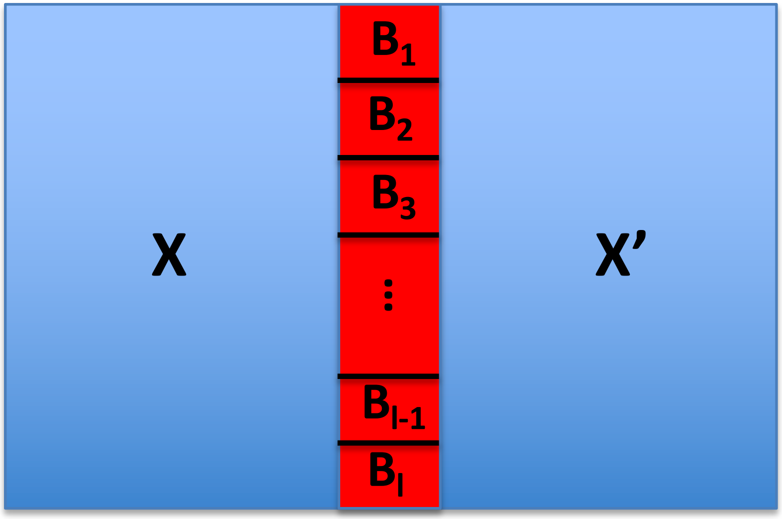
$\gamma = 0 \implies$ Local "boundary Hamiltonian"

$\gamma \neq 0 \implies$ Non-local "boundary Hamiltonian"

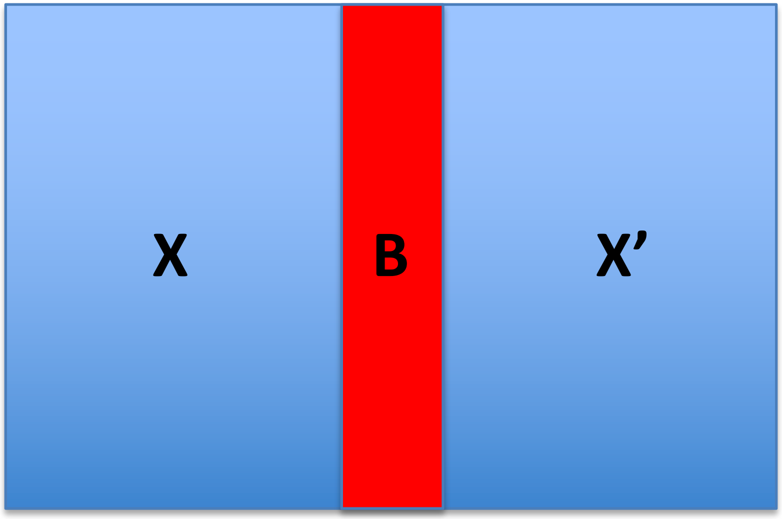
Result 2: Entanglement Spectrum

claim 2 Suppose $|\psi\rangle$ satisfies the area law assumption with $\gamma = 0$. Then

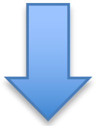
$$\lambda(\rho_X)^{\otimes 2} \approx \lambda(e^{\sum_k H_{B_k, B_{k+1}}})$$



From 1 to 2



$$
\begin{aligned}
& I(X : X') \\
= \quad & I(X : X'|B) \approx 0
\end{aligned}
$$



$$\rho_{XX'} \approx \rho_X \otimes \rho_{X'}$$

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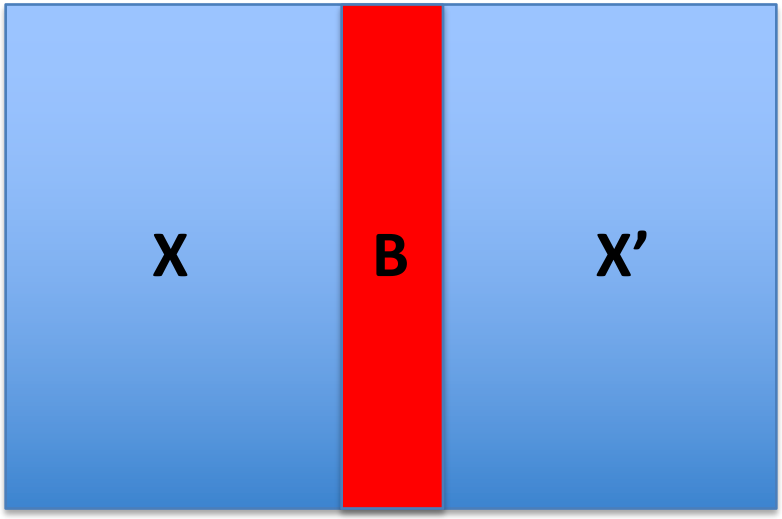

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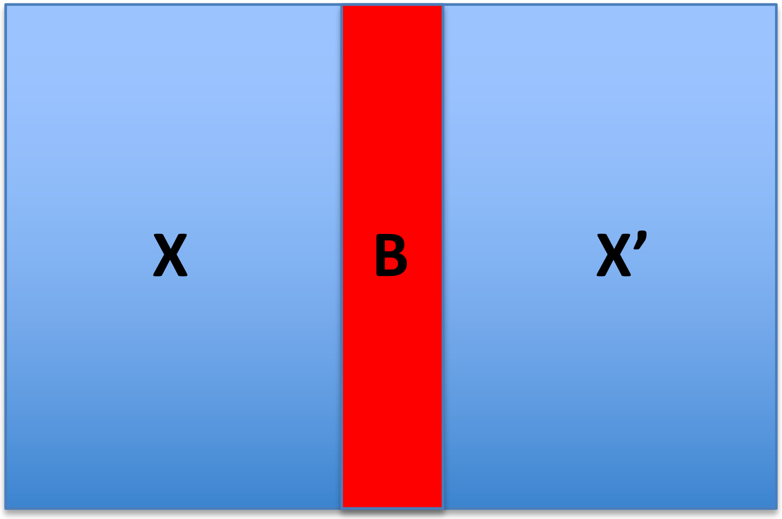
$$\lambda(\rho_{XX'}) = \lambda(\rho_B)$$

since $|\psi\rangle_{XBX'}$ is a pure state

From 1 to 2



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$$\Rightarrow \lambda(\rho_X) \otimes \lambda(\rho_{X'}) \approx \lambda(\rho_B)$$

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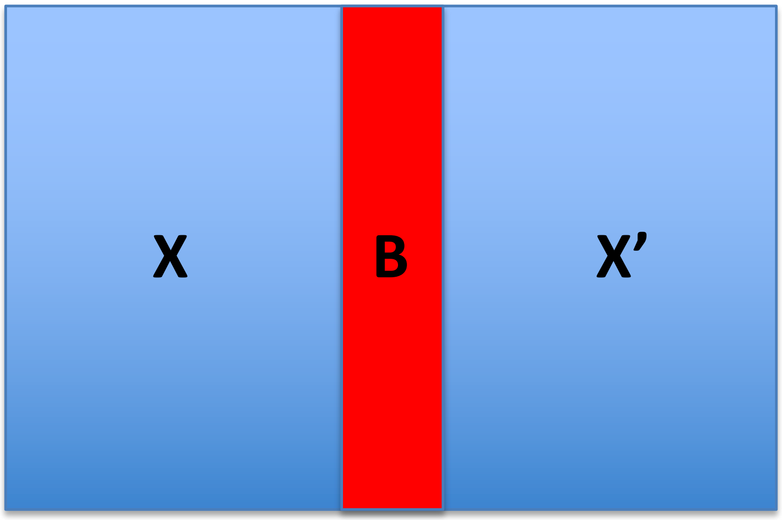

$$I(X : X') = I(X : X'|B) \approx 0$$

$$\Downarrow$$

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If $\gamma = 0, \quad \rho_B \approx e^{\sum_k H_{B_k, B_{k+1}}} / Z$

How to prove claim 1?

Let's focus on second part. Recap:

Suppose $\gamma = 0$. Then there is a local $H = \sum_k H_{B_k, B_{k+1}}$ s.t.

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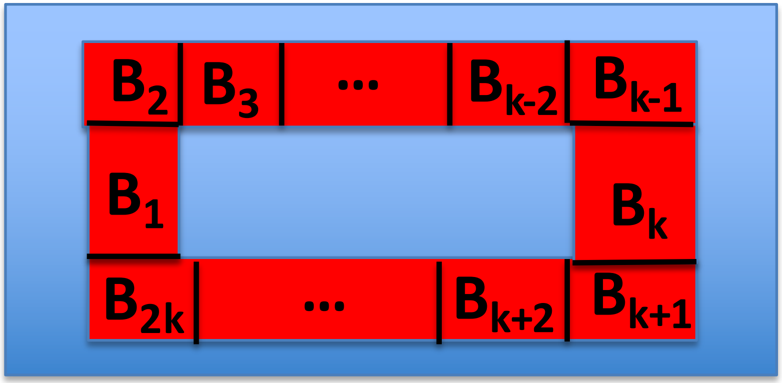

By area law: $I(B_1 \ldots B_i : B_{i+1} \ldots B_{2k-1} | B_i B_{2k}) \approx 0 \;\; \forall i$

By main theorem, we know it's thermal

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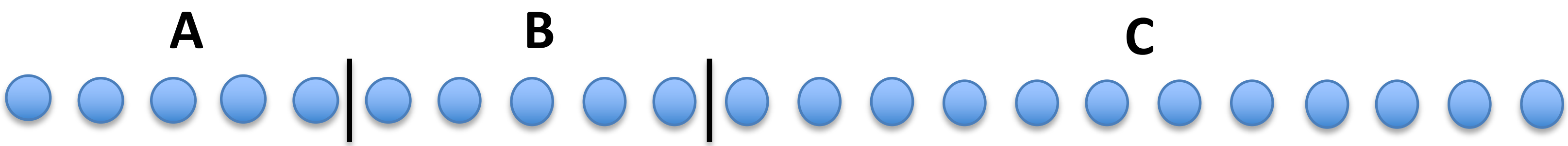
Apply part 2 of thm to get:

$$S(\rho_{B_1 \dots B_l} \| \exp(H)/Z) \leq \frac{n}{m} e^{-cm}$$

with l = n/m. Choose m = O(log(n)) to make error small

Recap main theorem



thm

1. Let *H* be a local Hamiltonian on *n* qubits. Then

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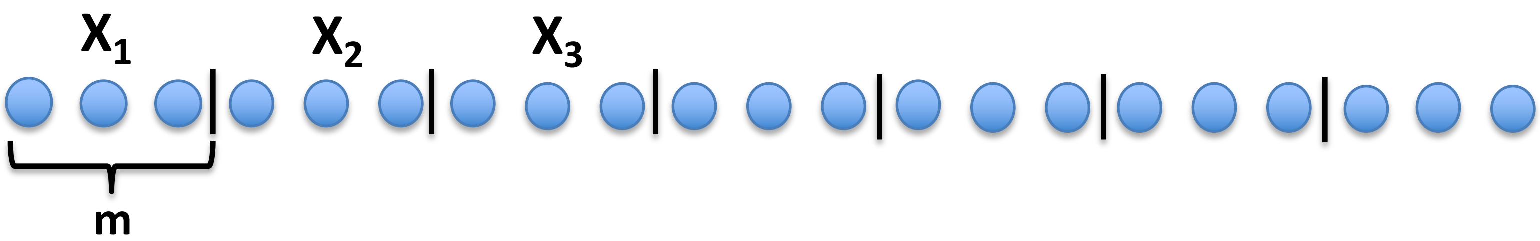
Proof Part 2



Let $\sigma_{X_1 \dots X_{\frac{n}{m}}}$ be the maximum entropy state s.t.

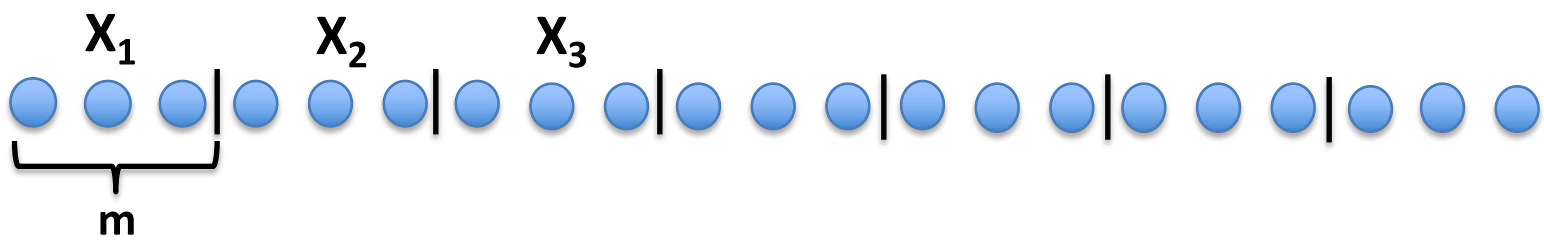
$$\sigma_{X_i, X_{i+1}} = \rho_{X_i, X_{i+1}} \quad \forall i \in [n/m]$$

Fact 1 (Jaynes' Principle): $\sigma = e^{\sum_k H_{X_k, X_{k+1}}}$

Fact 2 $$\min_{H \in \mathcal{H}_{2m}} S(\rho \| e^H / Z) \leq -S(\rho) - \mathrm{tr}(\rho \log \sigma)$$
$$= S(\sigma) - S(\rho)$$

Let's show it's small

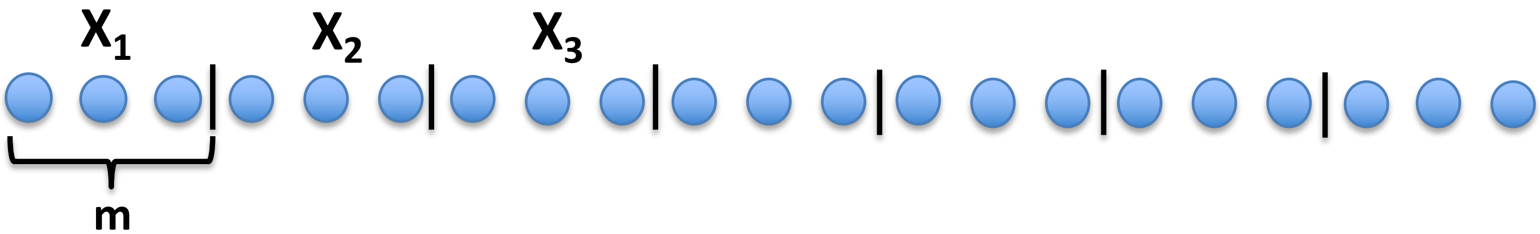
Proof Part 2



$$
\begin{aligned}
& S(X_1 \dots X_{n/m})_\sigma \\
\leq \quad & S(X_1 X_2)_\sigma - S(X_2)_\sigma + S(X_2 \dots X_{n/m})_\sigma
\end{aligned}
$$

SSA

Proof Part 2



$$
\begin{aligned}
& S(X_1 \ldots X_{n/m})_\sigma \\
\leq\ & S(X_1X_2)_\sigma - S(X_2)_\sigma + S(X_2 \ldots X_{n/m})_\sigma \\
\leq\ & S(X_1X_2)_\sigma - S(X_2)_\sigma + S(X_2X_3)_\sigma - S(X_3)_\sigma + S(X_3 \ldots X_{n/m})_\sigma
\end{aligned}
$$

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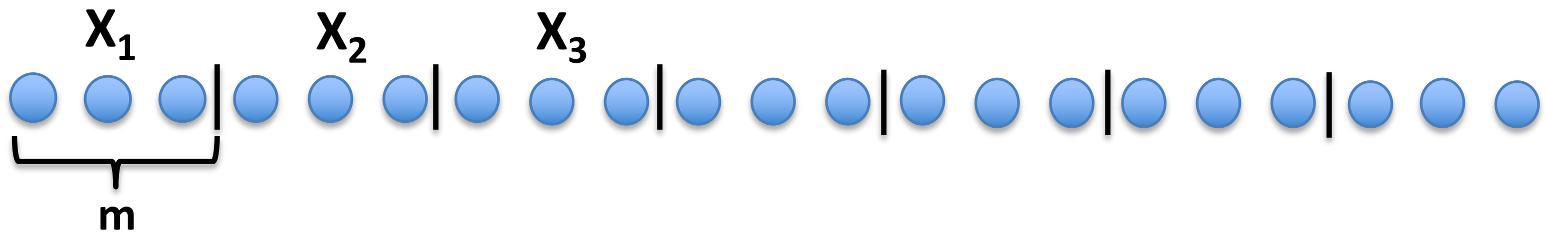

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\leq \; & \sum_i S(X_i X_{i+1})_\sigma - S(X_{i+1})_\sigma
\end{aligned}
$$

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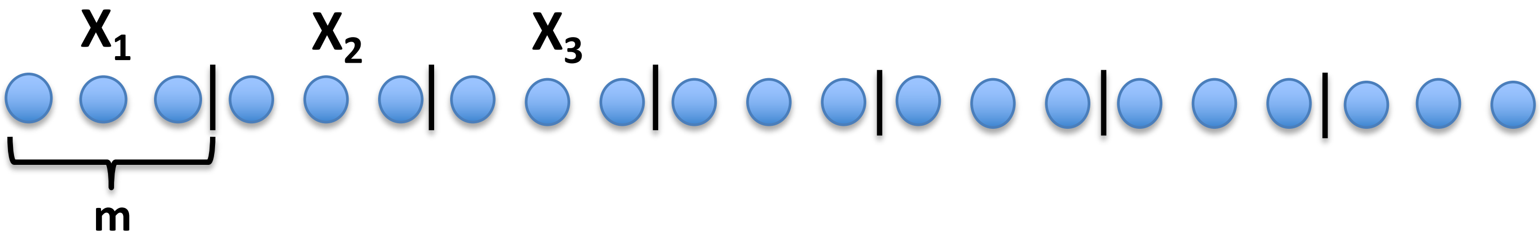

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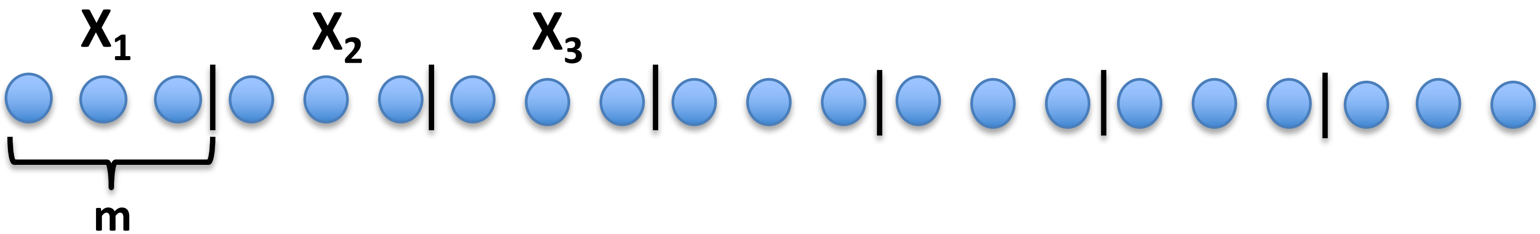
=\ & \sum_i S(X_i X_{i+1})_\rho - S(X_{i+1})_\rho
\end{aligned}
$$

Since $\sigma_{X_i, X_{i+1}} = \rho_{X_i, X_{i+1}} \quad \forall i \in [n/m]$

Proof Part 2



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= \quad & \sum_i S(X_i X_{i+1})_\rho - S(X_{i+1})_\rho \\
\leq \quad & S(X_1 \ldots X_{n/m})_\rho + \varepsilon \frac{n}{m}
\end{aligned}
$$

Since $I(X_i : X_{i+2} \ldots X_{n/m} | X_{i+1}) \leq \varepsilon \ \forall i$

Proof Part 1

Recap: Let *H* be a local Hamiltonian on *n* qubits. Then

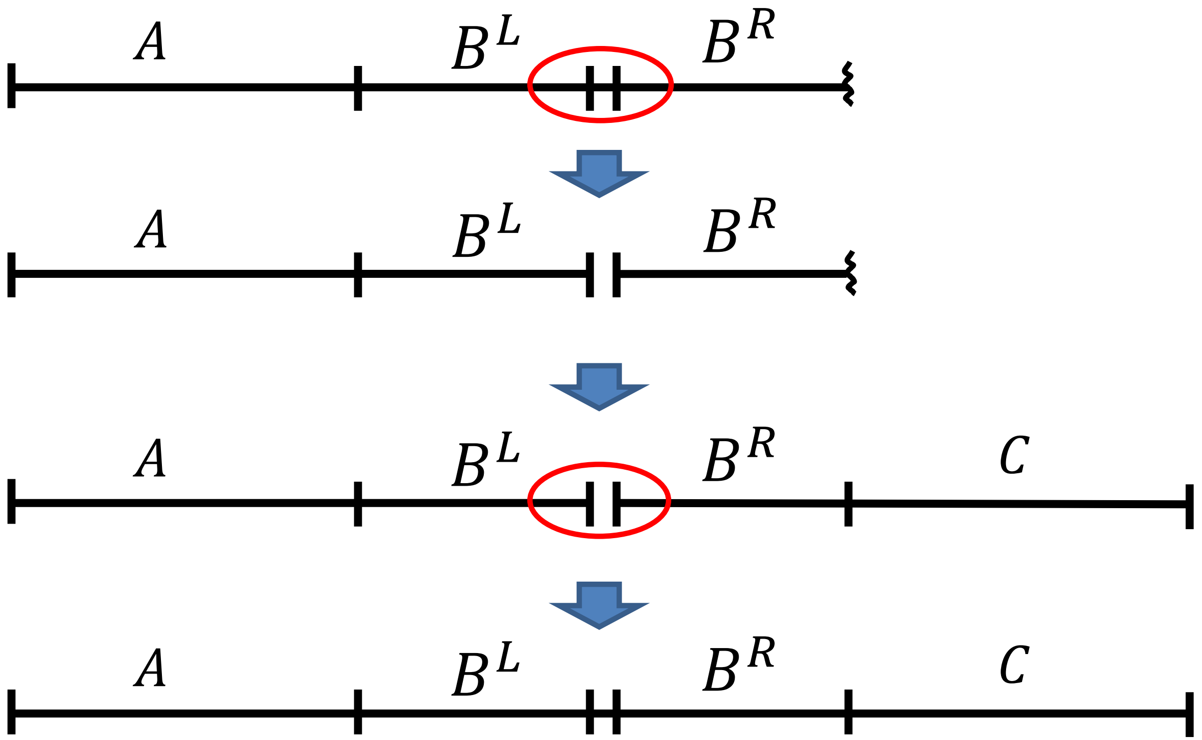
$$I(A:C|B)_{\rho_T} \leq e^{-c'\sqrt{|B|}+e^{c/T}}$$

We show there is a recovery channel from B to BC reconstructing the state on ABC from its reduction on AB.

Structure of Recovery Map

There exists an operator X_B such that

$$\rho^{H_{ABC}} \approx \mathrm{id}_A \otimes \kappa_{B\to \mathrm{BC}}\left(\rho_{AB}^{H_{ABC}}\right) = X_B\left(\mathrm{tr}_{B^R}\left[X_B^{-1}\rho_{AB}^{H_{ABC}}(X_B^{-1})^\dagger\right] \otimes \rho^{H_{B^R C}}\right)X_B^\dagger$$



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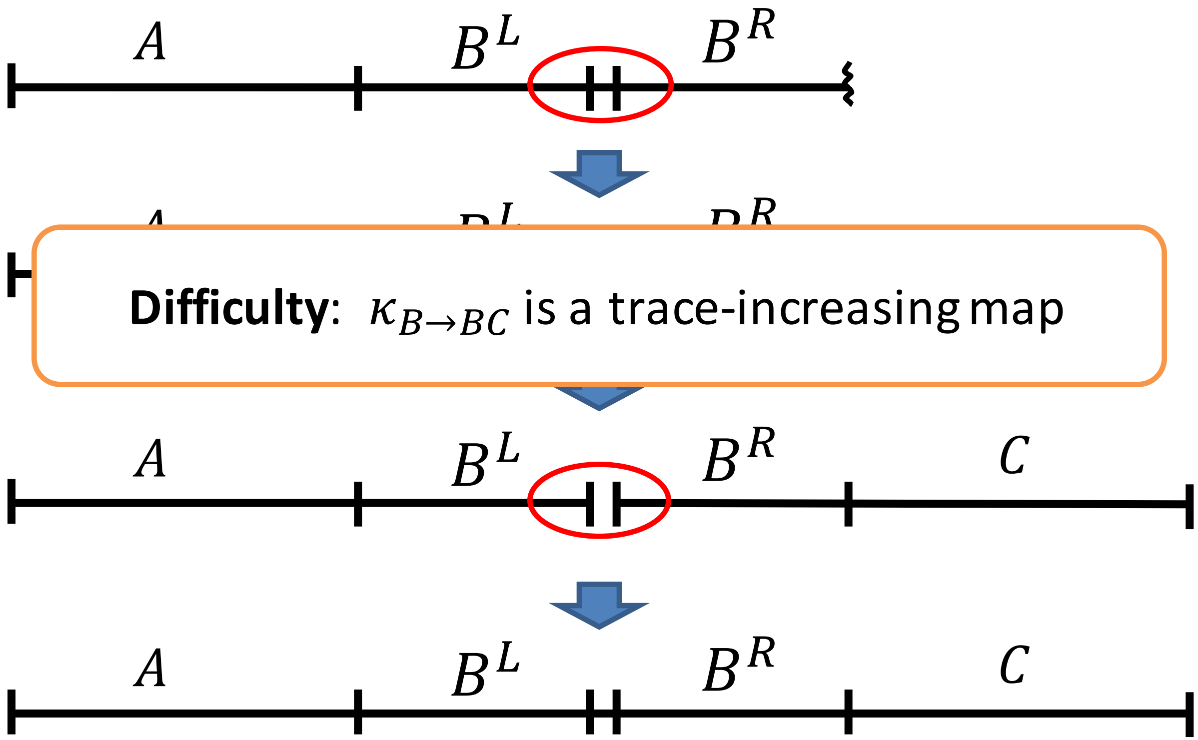
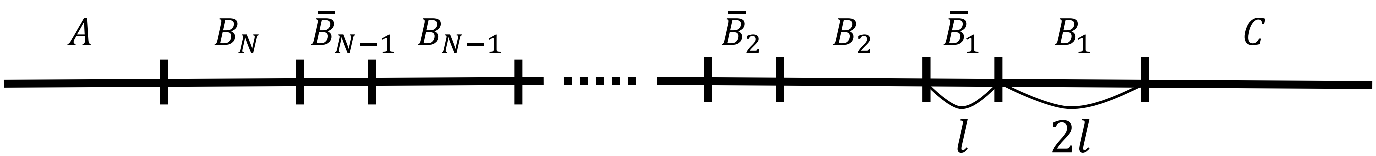
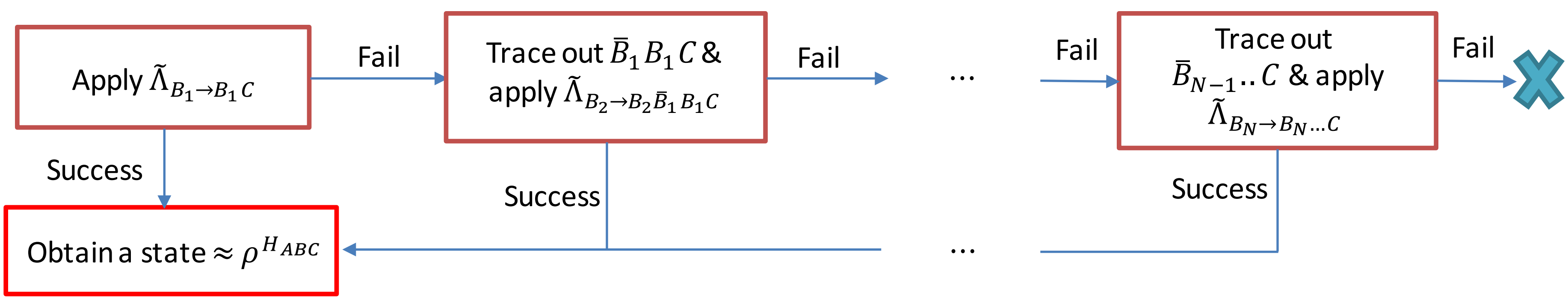


Difficulty: $\kappa_{B\to BC}$ is a trace-increasing map

Repeat-until-success Method

We normalize $\kappa_{B\to BC}$ and define a CPTD-map $\widetilde{\Lambda}_{B\to BC}$.
→ Succeed to recover with a constant probability p.





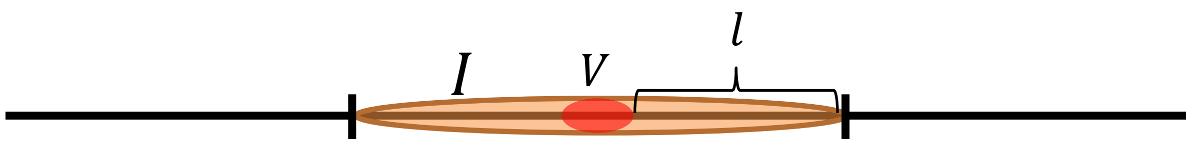
❑ Choose $N \sim l$ $\left(|B| = \mathcal{O}(l^2)\right)$.

→Total error=Fail probability $(1-p)^l$ + approx. error $\mathcal{O}\left(e^{-\mathcal{O}(l)}\right) = \mathcal{O}\left(e^{-\mathcal{O}(l)}\right)$.

Locality of Perturbations

The key point in the proof:

For a short-ranged Hamiltonian H, the local perturbation to H only perturb the Gibbs state locally.



A useful lemma by Araki (Araki, '69)
For 1D Hamiltonian with short-range interaction H,

$$\|e^{H+V}e^{-H} - e^{H_I+V}e^{-H_I}\| \leq \mathcal{O}\left(e^{-\mathcal{O}(l)}\right)$$

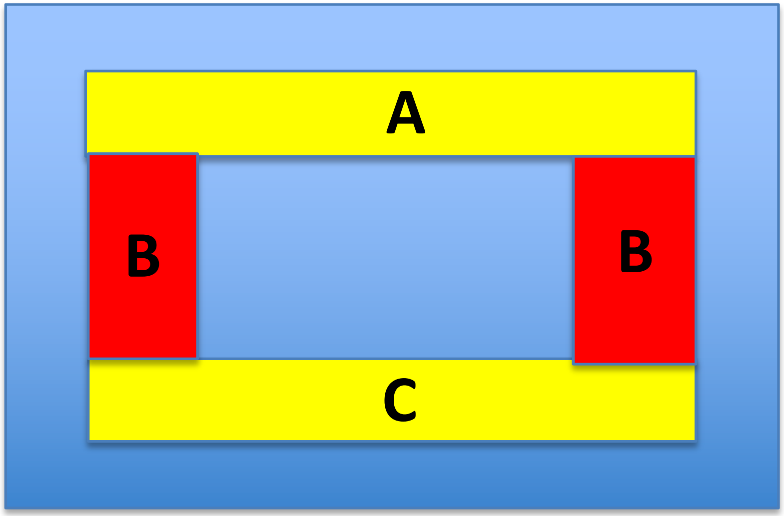
$$e^{-\beta H} \to e^{-\beta(H+V)} \approx X_I e^{-\beta H} X_I^{\dagger}$$

$$X_I = e^{-\frac{\beta}{2}(H_I+V)} e^{\frac{\beta}{2}H_I} \leftarrow \text{Local}$$

Proof claim 1 part 1

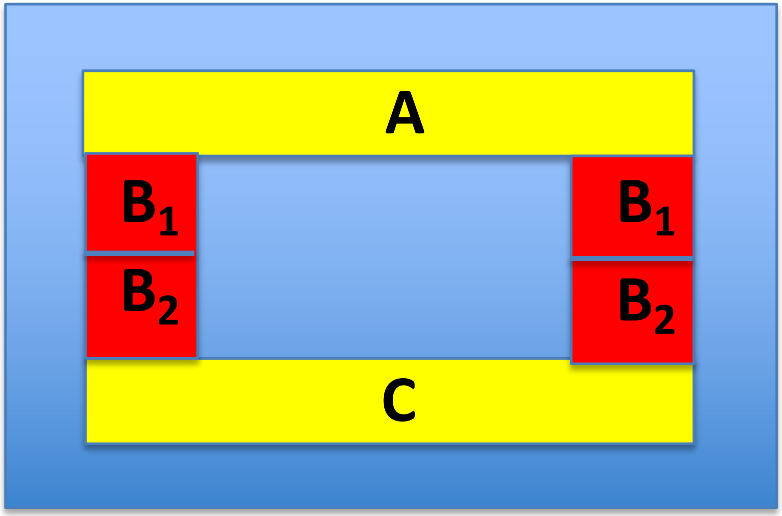
thm 1 Suppose $|\psi\rangle$ satisfies the area law assumption. Then

$$
\begin{aligned}
2\gamma &\approx I(A:C|B) \\
&\approx \min_{H_{AB},H_{BC}} S(\rho_{ABC} \| \exp(H_{AB}+H_{BC})/Z)
\end{aligned}
$$



Proof claim 1 part 1

We follow the strategy of (Kato et al '15) for the zero-correlation length case



Area Law implies

$$I(A : B_2|B_1) \approx 0$$

$$I(C : B_1|B_2) \approx 0$$

By Fawzi-Renner Bound, there are channels $\Lambda : B_1 \to B_1 A$, $\Delta : B_2 \to B_2 C$ s.t.

$$\Lambda(\rho_{B_1 B_2}) \approx \rho_{AB_1B_2}, \quad \Delta(\rho_{B_1B_2}) \approx \rho_{B_1B_2C}$$

Proof claim 1 part 1

Define: $\sigma_{AB_1B_2C} := \Lambda^{B_1 \to B_1 A} \otimes \Delta^{B_2 \to B_2 C}(\rho_{B_1 B_2})$

We have $\rho_{AB} \approx \sigma_{AB}, \ \rho_{BC} \approx \sigma_{BC}$

It follows that C can be reconstructed from B. Therefore

$$I(A:C|B)_\sigma \approx 0$$

Proof claim 1 part 1

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We have $\rho_{AB} \approx \sigma_{AB}, \; \rho_{BC} \approx \sigma_{BC}$

It follows that C can be reconstructed from B. Therefore

$$I(A:C|B)_\sigma \approx 0$$

Since

$$I(A:C|B)_\sigma = S(\sigma_{ABC} \| \exp(\log(\sigma_{AB})) + \log(\sigma_{BC})) - \log(\sigma_B)))$$

$\pi \approx \sigma$ with

$$\pi := \exp(\log(\sigma_{AB}) + \log(\sigma_{BC}) - \log(\sigma_B))/\mathrm{tr}(\ldots)$$

So $I(A:C|B)_\pi \approx 0$

Proof claim 1 part 1

Since $I(A:C|B)_\pi \approx 0$

$$\begin{aligned} S(ABC)_\pi &\approx S(AB)_\pi + S(BC)_\pi - S(B)_\pi \\ &\approx S(AB)_\rho + S(BC)_\rho - S(B)_\rho \\ &= S(ABC)_\rho + I(A:C|B)_\rho \end{aligned}$$

Let R_2 be the set of Gibbs states of Hamiltonians $H = H_{AB} + H_{BC}$. Then

$$\begin{aligned} \min_{\nu \in R_2} S(\rho\|\nu) &= \min_{\nu \in R_2} -S(\rho) - \mathrm{tr}(\rho \log \nu) \\ &\approx I(A:C|B)_\rho + \min_{\nu \in R_2} -S(\pi) - \mathrm{tr}(\rho \log \nu) \\ &\approx I(A:C|B)_\rho + \min_{\nu \in R_2} -S(\pi) - \mathrm{tr}(\pi \log \nu) \\ &= I(A:C|B)_\rho \end{aligned}$$

Summary

- Quantum Approximate Markov Chains are Thermal
- The double of the entanglement spectrum of topological trivial states is local

Open Problems:

- What happens in dims bigger than 2? Solve the conjecture!
- Are two copies of the entanglement spectrum necessary to get a local boundary model?
- What can we say about topological non-trivial states?

Thanks!