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1 Correcting a Shift [10 points]

Operators acting on a d-level quantum system (or *qudit*) can be expanded in terms of the d^2 "Pauli operators"

$$X^a Z^b,\ a, b = 0, 1, 2, ..., d-1. \tag{1}$$

Here X and Z are generalizations of the Pauli matrices σ_x and σ_z, which act in a particular basis $\{|j\rangle\}_{j=0}^{d-1}$ according to

$$X \ : \ |j\rangle \to |j+1\ (mod\ d)\rangle\,, \tag{2}$$

$$Z \ : \ |j\rangle \to \omega^j\,|j\rangle\,, \tag{3}$$

where $\omega = \exp(2\pi i/d)$. Note that it follows that

$$ZX = \omega XZ. \tag{4}$$

An error acting on a qudit can be expanded in this basis.

Suppose that errors with $|a|$ and $|b|$ small compared to d are common, but errors with large $|a|$ and $|b|$ are rare. We wish to design a quantum error-correcting code that corrects these small "shifts" in the amplitude or phase for the qudit.

For $d = nr_1r_2$ (where r_1 and r_2 are positive integers), consider the stabilizer generators

$$M_X = X^{nr_1},\ M_Z = Z^{nr_2} \tag{5}$$

(a) Verify that the generators commute.

(b) Find the commutation relations of M_X and M_Z with X^aZ^b.

(c) Find two generators of the normalizer. What commutation relations are satisfied by these normalizer generators? What is the dimension of the code subspace?

(d) How large an amplitude shift $|a|$ and phase shift $|b|$ can be corrected by this code?

2 Probabilistic Implementation of Gates by Cyclic Permutation and Measurement [10 Points]

Given an n-qubit Pauli operator $P = P_1 \otimes \cdots \otimes P_n$, $P_j \in \{I, X, Y, Z\}$, let $\eta(P) = P_2 \otimes P_3 \otimes \cdots \otimes P_n \otimes P_1$ be the Pauli operator obtained by cyclically permuting the factors. Let S be a stabilizer with generators $\{M_j\}_j$, and let $\{\bar{X}_\ell, \bar{Z}_\ell\}_\ell$ be generators of $N(S)/S$. Consider the stabilizer $\eta(S)$ defined by generators $\{M'_j := \eta(M_j)\}_j$, and the associated generators $\{\bar{X}'_\ell = \eta(\bar{X}_\ell), \bar{Z}'_\ell = \eta(\bar{Z})_\ell\}_\ell$ of $N(S)/S$.

Let us consider the following process:

- start with some encoded state $|\overline{\Psi}\rangle$ for S.
- measure all the stabilizer generators of $\eta(S)$.
- output 'success' and the post-measurement state if all outcomes are $+1$.

1. Let S be the stabilizer of the $[[5,1,3]]$-code. Compute, for an arbitrary encoded state $|\overline{\Psi}\rangle$, (a) the probability of success and (b) the resulting final state in the case of success.
2. Consider the set of stabilizers S and generators of $N(S)/S$ given by

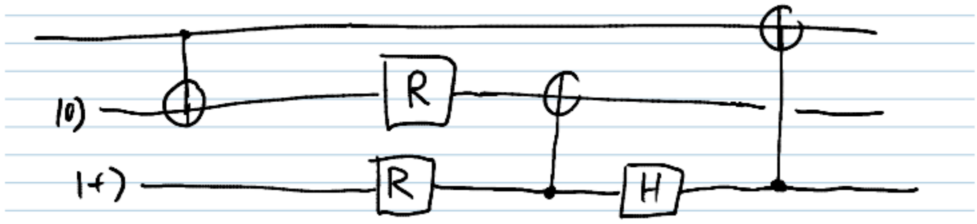
$$
\begin{array}{ccccccc}
M_1 & = & X & X & X & I \\
M_2 & = & I & Z & Z & I \\
\hline
\\
\bar{X}_1 & = & I & X & X & I \\
\bar{Z}_1 & = & Z & I & Z & I \\
\bar{X}_2 & = & I & I & I & X \\
\bar{Z}_2 & = & I & I & I & Z
\end{array}
$$

Suppose the initial state is the encoded state $|\overline{\Psi}\rangle = |\overline{+}\rangle \otimes |\overline{0}\rangle$. What is the probability of succeeding? Compute the resulting state (in the case of success) and show that it is entangled (between logical qubits).

Which logical gates were applied in the case of success?

3 Clifford Circuits for Stabilizer Codes [10 Points]

Consider the quantum code defined by the following encoding circuit



where where $R = diag(1, i)$. Find a set of stabilizer generators and a set of generators of $N(S)/S$.

4 Transversal Gates [10 Points]

(a) Consider the permutation-invariant code $\mathcal{Q}_n \subset (\mathbb{C}^2)^{\otimes n}$ from the previous problem set. Find a subcode $\mathbb{C}^2 \subset \mathcal{Q}_n$ encoding a qubit such that the unitary (logical) gate $\begin{pmatrix} 2^{-n/2} & \sqrt{1-2^{-n}} \\ \sqrt{1-2^{-n}} & 2^{-n/2} \end{pmatrix}$ is transveral (hint: "Put it in H"! (actual hint: use H)). (Remark: this is not fault-tolerant!)

(b) Let H_1 and H_2 be parity check matrices such that $\mathcal{Q} = CSS(H_1, H_2)$ is an $[[n, k, d]]$ CSS code. Find a necessary and sufficient condition on H_1 and H_2 such that $H^{\otimes n}$ preserves the code space (i.e.) $H^{\otimes n}\mathcal{Q} = \mathcal{Q}$ (where H is Hadamard). Check that your condition is satisfied if $H_1 = H_2$.

(c) Suppose that we take the code in (b) with $H = H_1 = H_2$ and n odd. Let $row(H)$ be the space spanned by the rows of H and assume that $|v| = \sum_{j=1}^{n} v_j = 0 \ mod \ 2$ for every $v \in \ row(H)$. Note that this implies $\vec{1} \in row(H)^{\perp} \backslash row(H)$ where $\vec{1} = (1, ., , , 1)$ (n-times). Argue that for a certain choice of encoded qubits, $H^{\otimes n}$ implements a logical unitary of the form $H \otimes C$, where C is a Clifford operation on $k - 1$ qubits.

(d) For the same code as in (c), argue that there is a transversal operation that implements a logical unitary of the form $R \otimes C$, where $R = diag(1, i)$ and C is a Clifford operator on $k - 1$ qubits.
