Physics 125b

Problem Set 5, Due Wednesday March 7, 2018

Problem 1

The trace of a linear operator A acting on a Hilbert space H is defined as

$$\mathrm{tr}(A) := \sum_{|i\rangle \in H} \langle i|A|i\rangle, \tag{1}$$

where the sum ranges over an orthornormal basis for H. Show that

- the trace is independent of the choice of orthornormal basis.
- the trace is cyclic, i.e. $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ for any two matrices A, B.
- If A is Hermitian, $\mathrm{tr}(A)$ gives the sum of the eigenvalues of A.

Problem 2

Show that if ρ_{SE} us pure then the non-zero eigenvalues of its reduced density matrices ρ_S and ρ_E and the corresponding degeneracies are the same.

Problem 3

Given a density operator ρ acting on H_S, what is the minimal dimension of the auxiliary Hilbert space H_E such that there exists a purification of ρ acting on $H_S \otimes H_E$? (Recall that the ensemble interpretations of any given density operator is not unique)

Problem 4

Imagine a qubit source that emits either of the two states $|0\rangle$ and $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$ with equal probability 1/2. Your task is to design a measurement scheme that allows to optimally distinguish these two cases. Unfortunately, the states $|0\rangle$ and $|+\rangle$ are not orthogonal, so you know that this cannot be done perfectly. Suppose now that your measurement scheme is not allowed to ever give a wrong answer. Instead, it is allowed to report one of three possible answers: that the true state is $|0\rangle$, that the true state is $|+\rangle$, or that the measurement outcome is inconclusive. We define the success probability of such a scheme as the probability that you identify the true state correctly.

- Show that for projective measurements the success probability is at most $1/4$.
- Find a POVM measurement that achieves a success probability strictly larger than $1/4$.