Physics 125b Solutions of Problem Set 6

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Problem 1

It will be easier to start from the second property:

$$\mathrm{tr}(AB) = \sum_i \langle i|AB|i\rangle = \sum_{i,j} \langle i|A|j\rangle\langle j|B|i\rangle = \sum_{i,j} \langle j|B|i\rangle\langle i|A|j\rangle = \sum_j \langle j|BA|j\rangle = \mathrm{tr}(BA) \tag{1}$$

Then we can apply above result, considering a new basis $|i'\rangle = U|i\rangle$ with a unitary matrix U such that:

$$\mathrm{tr}'(A) = \sum_{i'} \langle i'|A|i'\rangle = \sum_i \langle i|U^\dagger AU|i\rangle = \sum_i \langle i|UU^\dagger A|i\rangle = \sum_i \langle i|A|i\rangle = \mathrm{tr}(A) \tag{2}$$

Last, we for an operator A we can diagonalize it using some unitary operator by $A = U^\dagger DU$ then:

$$\mathrm{tr}(A) = \mathrm{tr}(U^\dagger DU) = \mathrm{tr}(D) = \text{sum of eigenvalues.} \tag{3}$$

Problem 2

Any state in Hilbert space $\mathcal{H}_{SE}$ can be written in terms of basis $|\Phi\rangle_{SE} = \sum_i \chi_i |\phi_i\rangle_S \otimes |\psi_i\rangle_E$ such that a pure density ρ_{SE} have the form:

$$\rho_{SE} = \left(\sum_i \chi_i |\phi_i\rangle_S \otimes |\psi_i\rangle_E\right)\left(\sum_j \chi_j^* \langle\phi_j|_S \otimes \langle\psi_j|_E\right) \tag{4}$$

Therefore their reduced densities are:

$$\begin{aligned}\rho_S &= \mathrm{tr}_E\left(\rho_{SE}\right) = \sum_i \langle\psi_i|_E \left(\sum_j \chi_j|\phi_j\rangle_S \otimes |\psi_j\rangle_E\right)\left(\sum_k \chi_k^*\langle\phi_k|_S \otimes \langle\psi_k|_E\right)|\psi_i\rangle_E \\ &= \sum_i |\chi_i|^2|\phi_i\rangle_E\langle\phi_i|_E \end{aligned} \quad (5)$$

and

$$\begin{aligned}\rho_E &= \mathrm{tr}_S\left(\rho_{SE}\right) = \sum_i \langle\phi_i|_S \left(\sum_j \chi_j|\phi_j\rangle_S \otimes |\psi_j\rangle_E\right)\left(\sum_k \chi_k^*\langle\phi_k|_S \otimes \langle\psi_k|_E\right)|\phi_i\rangle_S \\ &= \sum_i |\chi_i|^2|\psi_i\rangle_S\langle\psi_i|_S \end{aligned} \quad (6)$$

To compute their eigenvalue, it would better to express them by matrix:

$$\rho_S = \begin{pmatrix} |\chi_1|^2 & 0 & \cdots \\ 0 & |\chi_2|^2 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \quad \text{and} \quad \rho_E = \begin{pmatrix} |\chi_1|^2 & 0 & \cdots \\ 0 & |\chi_2|^2 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \quad (7)$$

Since their matrix forms are the same and both diagonal, the $|\chi_i|^2$'s are their eigenvalues. As a result, ρ_S ans ρ_E have the same eigenvalues and same corresponding degeneracies.

Problem 3

Given a Hilbert space H_S and an ensemble $\rho = \{p_i, |\phi_i\rangle | i = 1 \sim M\}$, we can find a purification $|\Phi\rangle_{SE} = \sum_{ij} A_{ij}|\phi_i\rangle_S \otimes \psi_j\rangle_E$ in $H_S \otimes H_E$ under the constrain :

$$\sum_{j=1\sim\dim(H_E)} A_{ij} \cdot A_{kj}^* = \delta_{ik} \cdot p_i. \quad (8)$$

For arbitrary $\{p_i\}$, this condition can be satisfied as long as $\dim(H_E) \geq M$. Following this argument, if we consider an ensemble $\tilde{\rho} = \{p_i, |\phi_i\rangle | i = 1 \sim \infty\}$, we would need a infinitely large auxiliary Hilbert space. Fortunately, as the hint says, the ensemble interpretation of any given density operator is not unique. For any ensemble with $M > \dim(H_S)$, we can always reduce the number of states down to $\dim(H_S)$ by singular value decomposition such that the meaningful largest M is bounded by $\dim(H_S)$. As a result, to cover purification of all possible density, we at least need an auxiliary Hilbert space with dimension $\dim(H_E) = \dim(H_S)$.

Problem 4

Projection Measurement

To make a projection measurement, we need to find a orthonormal basis which are eigenstates of our measurement. Now since our density is $\rho = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|+\rangle\langle +|$, we can either choose the basis as $\{|0\rangle, |1\rangle\}$ or $\{|+\rangle, |-\rangle\}$, such that if we take the measure and get $|1\rangle$ (or $|-\rangle$), then we can be sure that the input state must not be $|0\rangle$ (or $|+\rangle$), so we can identify the state without mistake. On the other side, if our measurement result is $|0\rangle$ (or $|+\rangle$), then the outcome is inconclusive because $\langle +|0\rangle \neq 0$. Therefore, the possibility to make an identification, when using basis $\{|0\rangle, |1\rangle\}$, without mistake is :

$$P(1) = \langle 1|\rho|1\rangle = \frac{1}{2}\langle 1|0\rangle\langle 0|1\rangle + \frac{1}{2}\langle 1|+\rangle\langle +|1\rangle = \frac{1}{4}. \tag{9}$$

Same result when basis $\{|+\rangle, |-\rangle\}$ is used:

$$P(-) = \langle -|\rho|-\rangle = \frac{1}{2}\langle -|0\rangle\langle 0|-\rangle + \frac{1}{2}\langle -|+\rangle\langle +|-\rangle = \frac{1}{4}. \tag{10}$$

POVM measurement

When we apply the POVM measurement, we don't need to choose the orthonormal basis as long as the operators are semi-positive defined. Following the logic in projective measurement, we want our final state of the measurement to be orthogonal to $|0\rangle$ or $|+\rangle$ so we construct following measuring operator:

$$\mathbb{E}_1 = C_1|1\rangle\langle 1| \text{ and } \mathbb{E}_- = C_-|-\rangle\langle -| \tag{11}$$

where $C_1 = C_- = C$ due to symmetry. Last, since we need to satisfy the condition $\sum_i \mathbb{E}_i = 1$, we add one more operator:

$$\mathbb{E} = \mathbb{I} - \mathbb{E}_1 - \mathbb{E}_- \tag{12}$$

whose eigenvalue can be obtained by solving:

$$\det \begin{pmatrix} 1 - C/2 - \lambda & C/2 \\ C/2 & 1 - 3C/2 - \lambda \end{pmatrix} = 0 \tag{13}$$

with basis $(|0\rangle, |1\rangle)^T$ whose solutions $\lambda = 1 - \left(1 \pm \frac{1}{\sqrt{2}}\right) C$ must be non-negative such that:

$$C \leq \frac{1}{1 + 1/\sqrt{2}} \tag{14}$$

Now we construct a measurement whose the final state can be $|1\rangle$, $|-\rangle$, or other. The possibility to give a non-mistake identification is:

$$P(1) + P(-) = C\langle 1|\rho|1\rangle + C\langle -|\rho|-\rangle = \frac{C}{2}. \tag{15}$$

The constant C can be chosen for $\frac{1}{2} \leq C \leq \frac{1}{1+1/\sqrt{2}}$ such that the success probability strictly larger than 1/4.