Physics 125b
Problem Set 5 Solutions

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Problem 1

We want to evaluate the Green's function

$$\begin{aligned} G^0(\vec{r}) &= \lim_{\epsilon\to 0}\int \frac{d^3q}{(2\pi)^3}\frac{e^{i\vec{q}\cdot\vec{r}}}{k^2-q^2-i\epsilon} \\ &= \lim_{\epsilon\to 0}\int_0^\infty dq\int_0^\pi d\theta\int_0^{2\pi} d\phi \frac{q^2\sin\theta}{(2\pi)^3}\frac{e^{iqr\cos\theta}}{k^2-q^2-i\epsilon} \\ &= \lim_{\epsilon\to 0}\frac{-i}{4\pi^2 k}\int_{-\infty}^\infty dq\frac{q}{k^2-q^2-i\epsilon}e^{-iqr} = \lim_{\epsilon\to 0}\frac{-i}{4\pi^2 k}\int dq f_\epsilon(q) \end{aligned}$$

Note that there are two (simple) poles satisfying $q^2 = k^2 - i\epsilon$, namely $q \simeq k - i\epsilon'$ and $q \simeq -k + i\epsilon'$. We choose the contour to be the upper semicircle. Using Jordan's lemma and the residue theorem, we obtain

$$G^0(\vec{r}) = \frac{-i}{4\pi^2 k}\lim_{\epsilon\to 0}\int dq f_\epsilon(q) = \frac{-i}{4\pi^2 k}\lim_{\epsilon\to 0}\left(2\pi i \ \mathrm{Res}_{f_\epsilon}(-k+i\epsilon')\right) = -\frac{e^{-ikr}}{4\pi r}.$$

The exponent has minus sign, indicating we have an incoming wave (compared to the outgoing solution obtained in class).

Problem 2

To solve the scattering problem, we want to solve the Schrödinger equation

$$-\frac{\hbar^2}{2m}\nabla^2\psi(\mathbf{x}) + V(\mathbf{x})\psi(\mathbf{x}) = E\psi(\mathbf{x}), \qquad (1)$$

with boundary conditions at infinity for the incoming and the scattered wave. Using spherical symmetry, we write

$$\psi(\mathbf{x}) = R(r)Y_{\ell m}(\theta, \phi), \tag{2}$$

and define $u(r) = rR(r)$, so we are only left with the radial Schrödinger equation,

$$u''(r) + k^2 u(r) = \left(\frac{\ell(\ell+1)}{r^2} + \frac{2m}{\hbar^2} V(r) \right) u(r), \tag{3}$$

to solve. Let's recall that k is defined by $E = \hbar^2 k^2 / 2m$. We are interested in the s-wave contribution, which means the $\ell = 0$ solution. The potential at hand is

$$V(r) = -\frac{\hbar^2}{ma^2} \frac{1}{\cosh^2(r/a)}. \tag{4}$$

Thus, the radial Schrödinger equation for this potential with $\ell = 0$ is,

$$u''(r) + k^2 u(r) + \frac{2}{a^2} \frac{1}{\cosh^2(r/a)} u(r) = 0, \tag{5}$$

which has the solutions

$$u(r) = e^{\pm ikr} (\tanh(r/a) \mp ika). \tag{6}$$

For the wavefunction $R(r) = \frac{u(r)}{r}$ to be nonsingular at $r = 0$, $u(r)$ needs to have a zero at $r = 0$. Thus, the solution $u(r)$ that satisfies this boundary condition is given by

$$u(r) = A \left(e^{ikr} (\tanh(r/a) - ika) + e^{-ikr} (\tanh(r/a) + ika) \right). \tag{7}$$

At $r \to \infty$, the wavefunction behaves as

$$R(r) \sim A \left(\frac{e^{ikr}}{r} (1 - ika) + \frac{e^{-ikr}}{r} (1 + ika) \right), \tag{8}$$

from which we can read off the scattering phase

$$e^{2i\delta_0} = -\frac{1 - ika}{1 + ika}. \tag{9}$$

The contribution to the total cross section is given by

$$\sigma_0 = \frac{4\pi}{k^2} \sin^2 \delta_0 = -\frac{4\pi}{k^2} \frac{e^{2i\delta_0} - 2 + e^{-2i\delta_0}}{4} \tag{10}$$

$$= \frac{4\pi}{k^2} \frac{1}{1 + k^2 a^2}. \tag{11}$$

Problem 3

The partial wave expansion is typically defined with the coefficients

$$f(\theta) = \sum_{\ell=0}^{\infty}(2\ell+1)a_\ell P_\ell(\cos\theta) \tag{12}$$

$$= \frac{1}{k}\sum_{\ell=0}^{\infty}(2\ell+1)e^{i\delta_\ell}\sin\delta_\ell\, P_\ell(\cos\theta). \tag{13}$$

For the given amplitude, only the partial waves for $\ell = 0$ and $\ell = 1$ are active, since only the corresponding $P_\ell(\cos\theta)$ appear, with coefficients

$$a_0 = \frac{\Gamma}{k_0 - k - ik\Gamma}\,, \quad a_1 = \frac{e^{2i\beta k^3}\sin(2\beta k^3)}{k}\,, \tag{14}$$

and corresponding phase shifts

$$\delta_0 = \arcsin\left(\frac{1}{\sqrt{1+(k_0-k)^2/k^2\Gamma^2}}\right)\,, \quad \delta_1 = 2\beta k^3. \tag{15}$$

Integrating the amplitude, we find the cross section

$$\sigma = \int_0^\pi d\theta \int_0^{2\pi} \sin\theta d\phi |f(\theta,\phi)|^2 \tag{16}$$

$$= \frac{2\pi}{k^2}\left(2\frac{k^2\Gamma^2}{(k_0-k)^2+k^2\Gamma^2} + \frac{2}{3}9\sin^2(2\beta k^3)\right) \tag{17}$$

$$= \frac{4\pi}{k^2}\left(\frac{k^2\Gamma^2}{(k_0-k)^2+k^2\Gamma^2} + 3\sin^2(2\beta k^3)\right). \tag{18}$$

The final form we have written the cross section in explicitly agrees with the optical theorem, $\sigma = \frac{4\pi}{k}\mathrm{Im} f(\theta = 0)$.