Physics 125b Solutions of Problem Set 2

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Problem 1

Compute the trial energy via integral

$$E_{trial} = \frac{\langle\phi|H|\phi\rangle}{\langle\phi|\phi\rangle} = \frac{\frac{\hbar^2 r_0}{m} + 2Zae^2(-a + e^{-\frac{2r_0}{a}}(a + r_0))}{2a^2 r_0} \tag{1}$$

with varying the parameter a, we solve for the minimum of trial energy with a_{min}, which satifies

$$\frac{\partial E_{trial}}{\partial a}|_{a=a_{min}} = 0 \tag{2}$$

one may expand a around $a_0 = \frac{\hbar^2}{me^2 Z}$ and solve for the linearized equation, or just do it numerically with Mathematica.... pluge a_{min} back into E_{trial} and compare with $E_0 = -\frac{me^4 Z^2}{2\hbar^2}$. By numerical computation with known physical constants (in a consistent unit!)

$$E_{trial} = -489.824eV \tag{3}$$

$$E_0 = -489.825eV \tag{4}$$

$$\frac{E_0 - E_{trial}}{E_0} \approx 1.7 \times 10^{-6} \tag{5}$$

The above result is suggested by doing everything numerically with Methematica, results should vary depend on how you do the approximation and the accuracy of the constants you use. But should be small anyway.

Problem 2

This is dimensional analysis.

$$n = \frac{\text{rate}}{\sigma c} = 4 \cdot 10^{28} m^{-3}. \tag{6}$$

Problem 3

From Eq. (19.3.8) in Shankar

$$\begin{aligned} \frac{d\sigma}{d\Omega} &= \left| \frac{2mV_0}{\hbar^2} \int \frac{\sin qr}{q} \theta(r_0 - r) r dr \right|^2 \qquad (7) \\ &= \frac{4m^2V_0^2}{\hbar^4} \frac{(\sin qr_0 - qr_0 \cos qr_0)^2}{q^6}. \qquad (8) \end{aligned}$$

The total cross-section is

$$\sigma = 2\pi \int_{\theta=0}^{\pi} \frac{d\sigma}{d\Omega} d(\cos\theta). \tag{9}$$

The trick is to write it in terms of q; since

$$q^2 = 2k^2(1 - \cos\theta), \tag{10}$$

we have

$$d(\cos\theta) = -\frac{qdq}{k^2} \tag{11}$$

so (up to a sign) the integral becomes

$$\begin{aligned} \sigma &= \frac{8\pi m^2 V_0^2}{\hbar^4 k^2} \int_0^{2k} \frac{(\sin qr_0 - qr_0 \cos qr_0)^2}{q^5} dq \qquad (12) \\ &= \frac{\pi m^2 V_0^2}{\hbar^4} \frac{32k^4 r_0^4 - 8k^2 r_0^2 + 4kr_0 \sin(4kr_0) + \cos(4kr_0) - 1}{16k^6}. \qquad (13) \end{aligned}$$

Sending $kr_0 \to 0$ is the same as sending $qr_0 \to 0$. Using that

$$\lim_{x \to 0} \frac{(\sin x - x\cos x)^2}{x^6} = \frac{1}{9} \tag{14}$$

we obtain

$$\frac{d\sigma}{d\Omega} = \frac{4m^2 V_0^2 r^6}{9\hbar^4}. \tag{15}$$

Since the θ dependence drops out (the scattering is isotropic) the total cross-section is obtained by multiplying with 4π.

Problem 4

(a) The differential cross section in the Born approximation is given by

$$\frac{d\sigma}{d\Omega} = (2\pi)^4 \mu^2 \hbar^2 |\langle p_f|V|p_i\rangle|^2.$$

Since the potential is a product of two terms, one depending on the position, the other — on the spin, $V = V_{position} V_{spin}$, we obtain

$$\begin{aligned} \frac{d\sigma}{d\Omega} &= (2\pi)^4 \mu^2 \hbar^2 |\langle p_f|V_{position}(\vec{r})|p_i\rangle|^2 |\langle \chi_f|V_{spin}|\chi_f\rangle|^2 \\ &= \frac{\mu^2}{4\pi^2\hbar^4} \left| \int d^3\vec{r} e^{-i\vec{q}\cdot\vec{r}} V(\vec{r}) \right|^2 |\langle \chi_f|V_{spin}|\chi_f\rangle|^2 \end{aligned}$$

(b) We have

$$|\chi_i\rangle = |\uparrow\downarrow\rangle = \frac{1}{\sqrt{2}} \left(\frac{|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle}{\sqrt{2}} + \frac{|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle}{\sqrt{2}} \right),$$

where the first state is the triplet state, and the second — the singlet. Also,

$$\vec{s}_a \cdot \vec{s}_b = \frac{1}{2}((\vec{s}_a + \vec{s}_b)^2 - (\vec{s}_a)^2 - (\vec{s}_b)^2).$$

We obtain

$$\begin{aligned} \vec{s}_a \cdot \vec{s}_b \frac{|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle}{\sqrt{2}} &= \frac{\hbar^2}{2} \left(2 - \frac{3}{4} - \frac{3}{4} \right) \frac{|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle}{\sqrt{2}}, \\ \vec{s}_a \cdot \vec{s}_b \frac{|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle}{\sqrt{2}} &= \frac{\hbar^2}{2} \left(0 - \frac{3}{4} - \frac{3}{4} \right) \frac{|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle}{\sqrt{2}}. \end{aligned}$$

Thus,

$$\vec{s}_a \cdot \vec{s}_b |\chi_i\rangle = \frac{\sqrt{5}\hbar^2}{4} \frac{-|\uparrow\downarrow\rangle + 2|\downarrow\uparrow\rangle}{\sqrt{5}} = \frac{\sqrt{5}\hbar^2}{4} |\chi'\rangle,$$

$$\sum_{\chi_f} |\langle \chi_f | V_{spin} | \chi_i \rangle|^2 = \sum_{\chi_f} |\langle \chi_f | \vec{s}_a \cdot \vec{s}_b | \chi_i \rangle|^2 = \frac{5\hbar^4}{16} \sum_{\chi_f} |\langle \chi_f | \chi' \rangle|^2 = \frac{5\hbar^4}{16}.$$

Note that since the summation is over some orthonormal basis (e.g. $|\uparrow\uparrow\rangle$, $|\uparrow\downarrow\rangle$, $|\downarrow\uparrow\rangle$, $|\downarrow\downarrow\rangle$), and $|\chi'\rangle\}$ is normalized, then the sum is equal to 1.

$$\begin{aligned} \frac{d\sigma}{d\Omega} &= \frac{\mu^2}{4\pi^2\hbar^4} \left| \int d^3\vec{r} e^{-i\vec{q}\cdot\vec{r}} V(\vec{r}) \right|^2 \left(\sum_{|\chi_f\rangle} |\langle \chi_f | V_{spin} | \chi_f \rangle|^2 \right) \\ &= \frac{5\mu^2}{64\pi^2} \left| \int d^3\vec{r} e^{-i\vec{q}\cdot\vec{r}} V(\vec{r}) \right|^2 = \frac{5\mu^2}{16q^2} \left| \int_{-\infty}^{\infty} dr e^{iqr} f(r) r \right|^2 \end{aligned}$$