Physics 125b Solutions of Problem Set 3

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Problem 1

Using first order time dependent perturbation theory, with the Hamiltonian given as:

$$H(t) = \frac{p^2}{2m} + \frac{1}{2} m\omega(t)^2 x^2 = \frac{p^2}{2m} + \frac{1}{2} m\omega_0^2 x^2 + \frac{\epsilon}{2} m\omega_0^2 \sin\left(\frac{\pi t}{T}\right) x^2 = H^{(0)} + \epsilon H^{(1)}(t), \tag{1}$$

we assume that the wave function have following form:

$$|\psi(t)\rangle = \sum_n d_n(t) e^{-iE_n^{(0)} t/\hbar} |n^{(0)}\rangle; \quad \text{where} \quad H^{(0)}|n^{(0)}\rangle = E_n^{(0)}|n^{(0)}\rangle. \tag{2}$$

with $E_n^{(0)} = \hbar\omega_0 \left(n + \frac{1}{2}\right)$. Putting this form back into Eq(1), we can get:

$$i\hbar \frac{\partial d_n(t)}{\partial t} = \sum_m d_m(t) e^{i\left(E_n^{(0)} - E_m^{(0)}\right) t/\hbar} \langle n^{(0)}|\epsilon H^{(1)}(t)|m^{(0)}\rangle \tag{3}$$

Now considering the expansion for $d_n(t)$ as:

$$d_n(t) = d_n^{(0)}(t) + \epsilon d_n^{(1)}(t) + \mathcal{O}(\epsilon^2) + \cdots \tag{4}$$

we can rewrite the Eq(3) by their ϵ orders:

$$\begin{aligned} i\hbar \frac{\partial d_n^{(0)}(t)}{\partial t} &= 0 \\ i\hbar \frac{\partial d_n^{(1)}(t)}{\partial t} &= \sum_m d_m^{(0)}(t) e^{i\left(E_n^{(0)} - E_m^{(0)}\right) t/\hbar} \langle n^{(0)}|H^{(1)}(t)|m^{(0)}\rangle \end{aligned} \tag{5}$$

Using the initial condition, $|\psi(t=0)\rangle = |0^{(0)}\rangle$, i.e. the ground state of unperturbated Hamiltonian, we can fix $d_n^{(0)}(t) = \delta_{n,0}$and put it into second line of Eq(5) to get:

$$i\hbar \frac{\partial d_n^{(1)}(t)}{\partial t} = e^{i\left(E_n^{(0)} - E_0^{(0)}\right)t/\hbar} \langle n^{(0)}|H^{(1)}(t)|0^{(0)}\rangle \tag{6}$$

First, we calculate $\langle n^{(0)}|x^2|0^{(0)}\rangle$ for the last term. Note that the x^2 can be rewritten in terms of the creation and annihilation operator:

$$x = \sqrt{\frac{\hbar}{2m\omega_0}} \left(a^\dagger + a\right), \tag{7}$$

so we have

$$\begin{aligned} \langle n^{(0)}|x^2|0^{(0)}\rangle &= \frac{\hbar}{2m\omega_0} \langle n^{(0)}|a^\dagger a^\dagger + a^\dagger a + a a^\dagger + aa|0^{(0)}\rangle \\ &= \frac{\hbar}{2m\omega_0} \left(\sqrt{2}\delta_{n,2} + \delta_{n,0}\right). \end{aligned} \tag{8}$$

As a result, the Eq(6) becomes:

$$i\hbar \frac{\partial d_n^{(1)}(t)}{\partial t} = e^{i\left(E_n^{(0)} - E_0^{(0)}\right)t/\hbar} \frac{\hbar\omega_0}{4} \sin\left(\frac{\pi t}{T}\right) \left(\sqrt{2}\delta_{n,2} + \delta_{n,0}\right) \tag{9}$$

For $n = 2$:

$$d_2^{(1)}(T) = \frac{\sqrt{2}\omega_0}{4i} \int_0^T e^{i\Delta E_2^{(0)} t/\hbar} \sin\left(\frac{\pi t}{T}\right) = \frac{i\omega_0 \left(1 + e^{i\left(\Delta E_2^{(0)}\right)T/\hbar}\right)}{2\sqrt{2}} \left(\frac{\frac{\pi}{T}}{\left(\frac{\Delta E_2^{(0)}}{\hbar}\right)^2 - \left(\frac{\pi}{T}\right)^2}\right), \tag{10}$$

where $\Delta E_2^{(0)} = 2\hbar\omega_0$, while $d_n^{(1)}(t) = 0$ for other excited states and we don't need the $d_0^{(1)}$. Since the $H(T) = H(0)$ and $|n^{(T)}\rangle = |n^{(0)}\rangle$, we can get the possibility to find the particle at excited state n as:

$$|\langle n^{(T)}|\psi(T)\rangle|^2 = |\epsilon d_2^{(1)}(T)|^2 = \frac{\epsilon^2\omega_0^2 \left(1 + \cos\left(2\omega_0 T\right)\right)}{4} \left(\frac{\frac{\pi}{T}}{(2\omega_0)^2 - \left(\frac{\pi}{T}\right)^2}\right)^2 \delta_{n,2} \tag{11}$$

Problem 2

Before going to the detail, we need to note that here our initial and final states are the ground state and the $E > 0$ propagating wave of the potential $V(r < r_0) = \frac{1}{2}m\omega^2(r^2 - r_0^2)$ and $V(r > r_0) = 0$. This is not really the harmonic potential we met before (so we can not use the $a/a^\dagger$). We'd discuss them respectively. As stated in the problem, approximating the ground state as the harmonic ground state is our first assumption. The reason to do this is that the wavefunction of the harmonic ground state drops as $\sim e^{-r^2}$ like a Gaussian, so as long as the r_0 cut is far enough, the wave function can't feel it at all. Quantitatively speaking, the energies of the wavefunction under these two potential are very close to each other, i.e.:

$$\langle 0|V_{\text{harmonic}}|0\rangle \approx \langle 0|V_{\text{harmonic with cutoff}}|0\rangle. \tag{12}$$

Consequently, when we adiabatically put in the cutoff, even though the wavefunction maintains, the energy will remain almost the same which validates our assumption. On the other hand, for a ionized state which can propagate to infinity, the potential sits only inside the $r < r_0$ region which contributes comparably infinitesimal energy and can be ignored. As a result, we can treat it as free plan wave of the $\sim e^{ikr}$ solution.

With the initial/final states, following the steps in Lecture Note 2 from p.34, we can write down transition matrix element but with new initial states:

$$\langle f^{(0)}|H^{(1)}|i^{(0)}\rangle = \frac{e}{2mc}\frac{1}{(2\pi\hbar)^{3/2}}\int d^3r e^{-i\vec{p}_f\cdot\vec{r}/\hbar}e^{i\vec{k}\cdot\vec{r}}\vec{A}_0\cdot\left(-i\hbar\vec{\nabla}\right)\left[\left(\frac{m\omega}{\pi\hbar}\right)^{3/4}e^{-\frac{m\omega r^2}{2\hbar}}\right] \tag{13}$$

Using the dipole approximation $e^{i\vec{k}}\cdot\vec{r}\approx 1$ and intergrating by parts with p_f chosen to point to z-axis, we can get:

$$\begin{aligned}
\langle f^{(0)}|H^{(1)}|i^{(0)}\rangle &= \frac{e}{2mc}\frac{1}{(2\pi\hbar)^{3/2}}\left(\frac{m\omega}{\pi\hbar}\right)^{3/4}\vec{p_f}\cdot\vec{A}_0\int e^{ip_f r\cos\theta/\hbar}e^{-\frac{m\omega r^2}{2\hbar}}r^2\sin\theta dr d\theta d\phi \\
&= \frac{e}{2mc}\frac{1}{(2\pi\hbar)^{3/2}}\left(\frac{m\omega}{\pi\hbar}\right)^{3/4}\vec{p_f}\cdot\vec{A}_0\frac{4\pi\hbar}{p_f}\int_0^\infty \sin\left(\frac{p_f r}{\hbar}\right)e^{m\omega r^2/2\hbar}r dr \\
&= \frac{e}{2mc}\left(\frac{1}{\pi m\omega\hbar}\right)^{3/4}\vec{p_f}\cdot\vec{A}_0\cdot\exp\left(\frac{-p_f^2}{2m\omega\hbar}\right)
\end{aligned} \tag{14}$$

As a result, the Fermi's golden rule gives:

$$R_{i\to f} = \frac{2\pi}{\hbar}\left(\frac{e}{2mc}\right)^2 \frac{1}{(\pi m\omega\hbar)^{3/2}} \left(\vec{p}_f\cdot\vec{A}_0\right)^2 \exp\left(\frac{-p_f^2}{m\omega\hbar}\right)\delta\left(E_f^0 - E_i^0 - \hbar\omega\right) \tag{15}$$

Last step is to sum over all possible final states:

$$\begin{aligned}
R_{i\to\text{all}} &= \int dp_f p_f^2 d\Omega R_{i\to f} \\
&= \frac{2\pi}{\hbar}\left(\frac{e}{2mc}\right)^2 \frac{1}{(\pi m\omega\hbar)^{3/2}} \int dp_f p_f^2 d\Omega \left(\vec{p}_f\cdot\vec{A}_0\right)^2 \exp\left(\frac{-p_f^2}{m\omega\hbar}\right)\delta\left(E_f^0 - E_i^0 - \hbar\omega\right) \\
&= \frac{2\pi}{\hbar}\left(\frac{e}{2mc}\right)^2 \frac{1}{(\pi m\omega\hbar)^{3/2}} \int dp_f p_f^2 d\Omega \left(p_f A_0\cos\theta\right)^2 \exp\left(\frac{-p_f^2}{m\omega\hbar}\right)\delta\left(\frac{p_f^2}{2m} - E_0 - \hbar\omega\right) \\
&= \frac{8\pi^2}{3\hbar}\left(\frac{e}{2mc}\right)^2 \frac{A_0^2}{(\pi m\omega\hbar)^{3/2}} \int dp_f p_f^2 \exp\left(\frac{-p_f^2}{m\omega\hbar}\right)\delta\left(\frac{p_f^2}{2m} - E_0 - \hbar\omega\right) \\
&= \frac{8\pi^2}{3\hbar}\left(\frac{e}{2mc}\right)^2 \frac{mA_0^2 p_0^3}{(\pi m\omega\hbar)^{3/2}} \exp\left(\frac{-p_0^2}{m\omega\hbar}\right)
\end{aligned} \tag{16}$$

where $p_0 = \sqrt{m\left(5\hbar\omega - m\omega^2 r_0^2\right)}$ which comes from the the delta function.