Physics 125b
Problem Set 2 Solutions

February 4, 2018

Problem 1

The perturbation breaks rotational invariance of the Hamiltonian. One can see this by e.g. evaluating the commutator $[J_x, H^{(1)}]$ and verifying that $[J_x, H^{(1)}] \neq 0$, which is evident by the explicit z dependence of $H^{(1)}$.

The eigenvectors of the unperturbed Hamiltonian will be indexed by the total energy of two 3D harmonic oscillators, E_{tot}. Note that 3D harmonic oscillator can be treated as three 1D oscillators, and thus the energy levels of fermion i can be labeled by a triple of numbers (n_i^x, n_i^y, n_i^z). Note that

$$z_i = \sqrt{\frac{\hbar}{2m\omega}}(a_i^z + (a_i^z)^\dagger) \tag{1}$$

When we evaluate first order corrections $\langle\psi|H^{(1)}|\psi\rangle$ to energy for the state $|\psi\rangle$ with total energy E_{tot}, the perturbation $H^{(1)}$ acting on $|\psi\rangle$ will create a superposition of states with different total energy, $E_{tot} \pm \hbar\omega$ (if $|\psi\rangle$ is the ground state, then only $E_{tot} + \hbar\omega$). We know that states with different energies are orthogonal and thus first order corrections vanish.

Another way to see this is to note that the first order shift in energy is determined by the expectation value of z_1 in the unperturbed states, but since the unperturbed Hamiltonian is spherically symmetric, these expectation values have to vanish, thus so does the first order energy shift.

Problem 2

Note that the wavefunction describing the system has to be antisymmetric. There is a unique ground state $|\psi_0\rangle$ with total energy $E_0 = 3\hbar\omega$, namely

$$|\psi_0\rangle = \frac{1}{\sqrt{2}}|0,0,0\rangle \otimes |0,0,0\rangle \otimes (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle). \tag{2}$$

There are 12 excited states $|\psi_i\rangle$ with total energy $E_i = 4\hbar\omega$, for which we can take the following orthonormal basis,

$$
\begin{aligned}
|\psi_1\rangle &= \frac{1}{2}(|1,0,0\rangle \otimes |0,0,0\rangle + |0,0,0\rangle \otimes |1,0,0\rangle) \otimes (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle), & (3)\\
|\psi_2\rangle &= \frac{1}{2}(|0,1,0\rangle \otimes |0,0,0\rangle + |0,0,0\rangle \otimes |0,1,0\rangle) \otimes (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle), & (4)\\
|\psi_3\rangle &= \frac{1}{2}(|0,0,1\rangle \otimes |0,0,0\rangle + |0,0,0\rangle \otimes |0,0,1\rangle) \otimes (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle), & (5)\\
|\psi_4\rangle &= \frac{1}{\sqrt{2}}(|1,0,0\rangle \otimes |0,0,0\rangle - |0,0,0\rangle \otimes |1,0,0\rangle) \otimes |\uparrow\uparrow\rangle, & (6)\\
|\psi_5\rangle &= \frac{1}{\sqrt{2}}(|0,1,0\rangle \otimes |0,0,0\rangle - |0,0,0\rangle \otimes |0,1,0\rangle) \otimes |\uparrow\uparrow\rangle, & (7)\\
|\psi_6\rangle &= \frac{1}{\sqrt{2}}(|0,0,1\rangle \otimes |0,0,0\rangle - |0,0,0\rangle \otimes |0,0,1\rangle) \otimes |\uparrow\uparrow\rangle, & (8)\\
|\psi_7\rangle &= \frac{1}{2}(|1,0,0\rangle \otimes |0,0,0\rangle - |0,0,0\rangle \otimes |1,0,0\rangle) \otimes (|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle), & (9)\\
|\psi_8\rangle &= \frac{1}{2}(|0,1,0\rangle \otimes |0,0,0\rangle - |0,0,0\rangle \otimes |0,1,0\rangle) \otimes (|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle), & (10)\\
|\psi_9\rangle &= \frac{1}{2}(|0,0,1\rangle \otimes |0,0,0\rangle - |0,0,0\rangle \otimes |0,0,1\rangle) \otimes (|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle), & (11)\\
|\psi_{10}\rangle &= \frac{1}{\sqrt{2}}(|1,0,0\rangle \otimes |0,0,0\rangle - |0,0,0\rangle \otimes |1,0,0\rangle) \otimes |\downarrow\downarrow\rangle, & (12)\\
|\psi_{11}\rangle &= \frac{1}{\sqrt{2}}(|0,1,0\rangle \otimes |0,0,0\rangle - |0,0,0\rangle \otimes |0,1,0\rangle) \otimes |\downarrow\downarrow\rangle, & (13)\\
|\psi_{12}\rangle &= \frac{1}{\sqrt{2}}(|0,0,1\rangle \otimes |0,0,0\rangle - |0,0,0\rangle \otimes |0,0,1\rangle) \otimes |\downarrow\downarrow\rangle. & (14)
\end{aligned}
$$

Note that

$$H^{(1)}|\psi_0\rangle = \frac{\epsilon\hbar}{2}\sqrt{\frac{\hbar}{4m\omega}}|0,0,1\rangle \otimes |0,0,0\rangle \otimes (-|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) + \tag{15}$$

$$\frac{\epsilon\hbar}{2}\sqrt{\frac{\hbar}{4m\omega}}|0,0,0\rangle \otimes |0,0,1\rangle \otimes (|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) = \tag{16}$$

$$\frac{\epsilon\hbar}{2}\sqrt{\frac{\hbar}{4m\omega}}(-|0,0,1\rangle \otimes |0,0,0\rangle + |0,0,0\rangle \otimes |0,0,1\rangle) \otimes (|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \tag{17}$$

Notice that this state is proportional to the eigenstate $|\psi_9\rangle$, and therefore orthogonal to every other unperturbed energy eigenstate, so the sum over states

$$\sum_{m\neq 0} \frac{|\langle m|H^{(1)}|\psi_0\rangle|^2}{E_0 - E_m} \tag{18}$$

collapses to a single term and we get

$$\Delta E_0 = \frac{|\langle\psi_9|H^{(1)}|\psi_0\rangle|^2}{E_0 - E_1} = -\frac{\epsilon^2\hbar^2}{4m\omega^2}. \tag{19}$$

Note that for Eq. (19) to be correct, the state $|\psi_9\rangle$ has to be normalized.

Problem 3

We have

$$p = i\sqrt{\frac{\hbar m\omega}{2}}(a^\dagger - a) \tag{20}$$

and we notice that $\langle k|\frac{\eta p}{m}|n\rangle$ is non-zero only for $k = n \pm 1$. We obtain

$$\langle n+1|\frac{\eta p}{m}|n\rangle = \frac{i\eta}{m}\sqrt{\frac{\hbar m\omega}{2}}\sqrt{n+1}, \tag{21}$$

$$\langle n-1|\frac{\eta p}{m}|n\rangle = -\frac{i\eta}{m}\sqrt{\frac{\hbar m\omega}{2}}\sqrt{n}, \tag{22}$$

so from the first-order time dependent perturbation theory we obtain the amplitudes of the state after time t

$$c_n(t) = 1, \quad (23)$$

$$c_{n+1}(t) = -\frac{i}{\hbar}\int_0^t \langle n+1|H^{(1)}|n\rangle e^{i\omega\tau}d\tau = \frac{\eta}{\hbar m}\sqrt{\frac{\hbar m\omega}{2}}\sqrt{n+1}\frac{1}{i\omega}(e^{i\omega t}-1), \quad (24)$$

$$c_{n-1}(t) = -\frac{\eta}{\hbar m}\sqrt{\frac{\hbar m\omega}{2}}\sqrt{n}\frac{1}{-i\omega}(e^{-i\omega t}-1). \quad (25)$$

The (normalized) wavefunction after time t is given by

$$|\psi(t)\rangle = |n\rangle + c_{n+1}(t)|n+1\rangle + c_{n-1}(t)|n-1\rangle + \mathcal{O}(\eta^2). \quad (26)$$

The probabilities to transit to state $|n+1\rangle$ and $|n-1\rangle$ are

$$P_{n+1}(t) = |c_{n+1}(t)|^2 \quad (27)$$

$$P_{n-1}(t) = |c_{n-1}(t)|^2, \quad (28)$$

From unitarity, these imply the probability to stay in $|n\rangle$ is $P_n(t) = 1 - P_{n+1}(t) - P_{n-1}(t)$. Another way to see this is normalizing the state $|\psi(t)\rangle$ to leading order in η. The change of energy to leading order in η is given by

$$\begin{aligned}
\Delta E &= \langle\psi(T)|H|\psi(T)\rangle - E_n && (29)\\
&= E_{n+1}P_{n+1}(T) + E_{n-1}P_{n-1}(T) + E_nP_n(T) - E_n && (30)\\
&= (E_{n+1}-E_n)P_{n+1}(T) + (E_{n-1}-E_n)P_{n-1}(T) && (31)\\
&= 2\frac{\eta^2}{m}\sin^2\frac{\omega T}{2}. && (32)
\end{aligned}$$

The rate is simply $\Delta E/T$.