Physics 125a - Problem Set 1 - due October 5, 2016.

Problem 1 **(5 points)**

Consider three elements from the vector space (over the real numbers) of real 2×2 matrices,

$$|1\rangle = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, |2\rangle = \begin{bmatrix} 2 & 2 \\ -2 & -2 \end{bmatrix}, |3\rangle = \begin{bmatrix} 2 & 2 \\ 0 & -2 \end{bmatrix} \tag{1}$$

Are they linearly independent? Justify you answer. Note that we are calling these matrices vector and using kets to represent them in order to emphasie their role as elements of a vector space.

Problem 2 **(10 points)**

Hermitian and unitary matrices have a complete basis of eigenvectors. This is not true for all matrices.

- **(a)** Find the eigenvalues of the matrix $\Lambda = \begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix}$ and show that all eigenvectors are proportional to $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.
- **(b)** Find the eigenvalues and orthornormal eigenvectors of the Hermitian matrix,

  $$\Omega = \frac{1}{2} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}.$$

Note: The above calculations should be done by hand, without the help of a computer.

Problem 3 **(10 points)**

Consider the Hermitian matrices M^1, M^2, M^3, M^4 that satisfy,

$$M^i M^j + M^j M^i = 2\delta_{ij} I, \quad i, j \in \{1, 2, 3, 4\}.$$

- **(a)** Show that the eigenvalues of the matrices M^i are ± 1.
  Hint: Go to the eigenbasis of M^i and use the equation above for $i = j$.
- **(b)** Show the matrices M^i are traceless.
  Hint: Use $M^i M^j = -M^j M^i$ for $i \neq j$ and recall that $\text{tr}(ABC) = \text{tr}(BCA)$.
- **(c)** Show that the matrices M^i cannot have odd dimension.

Problem 4 (5 points)

Argue that as $\Delta \to 0$ the function,

$$g_\Delta(x - x') = \frac{1}{|\Delta|\sqrt{\pi}} \exp\left[-\frac{(x - x')^2}{\Delta^2}\right]$$

approaches the Dirac delta function $\delta(x - x')$.

Problem 5 (10 points)

Let V be the space of real-valued, twice-differentiable functions $f(x)$, defined on the interval $x \in [0, 2\pi]$, such that

$$f(0) = f(2\pi) = 0.$$

- **(a)** Show that V is a vector space over the field of real numbers

  Define the inner product of $f, g \in V$ by

  $$\langle f|g\rangle = \int_0^{2\pi} f(x)g(x)dx.$$

- **(b)** Show that this definition obeys the axioms of an inner product.
  Consider the operator

  $$O = \frac{d^2}{dx^2}$$

- **(c)** Show that O is a Hermitian operator in the vector space V.
- **(d)** Write down the complete set of eigenfunctions $\{F_n(x)\}$ of the operator O and give their corresponding eigenvalues.
- **(e)** Show that, with an appropriate choice of normalization, they form an orthornormal basis, i.e.

  $$\langle F_n|F_m\rangle = \delta_{nm}.$$

  Indicsate the normalization for the F_n's.
- **(f)** Argue that any function $f(x) \in V$ may be expressed as a sum over the eigenfunctions of O:

  $$f(x) = \sum_n a_n F_n(x).$$

  Hint: You can just quote the relevant result from the theory of Fourier series.
- **(g)** Use the results of parts (e) and (f) to show that the norm of $f \in V$ can be expressed as

  $$|f| = \sqrt{\sum_n a_n^2}.$$