

Physics 125b Solutions of Problem Set 1

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Problem 1

For the first order energy shift of ground state, we need to compute

$$\langle 0|H^{(1)}|0\rangle = \langle 0|\epsilon\cos(kX)|0\rangle \quad (1)$$

use the operator basis with a a^\dagger

$$\cos(kX) = \cos\left(k\sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger)\right) = \frac{1}{2}\left(e^{ik\sqrt{\frac{\hbar}{2m\omega}}(a+a^\dagger)} + e^{-ik\sqrt{\frac{\hbar}{2m\omega}}(a+a^\dagger)}\right) \quad (2)$$

use the Baker Hausdorff formula with

$$[\pm ik\sqrt{\frac{\hbar}{2m\omega}}a^\dagger, \pm ik\sqrt{\frac{\hbar}{2m\omega}}a] = k^2\frac{\hbar}{2m\omega} \quad (3)$$

$$\langle 0|H^{(1)}|0\rangle = \frac{\epsilon}{2}e^{-k^2\frac{\hbar}{4m\omega}}\langle 0|e^{ik\sqrt{\frac{\hbar}{2m\omega}}a^\dagger}e^{ik\sqrt{\frac{\hbar}{2m\omega}}a} + e^{-ik\sqrt{\frac{\hbar}{2m\omega}}a^\dagger}e^{-ik\sqrt{\frac{\hbar}{2m\omega}}a}|0\rangle = \epsilon e^{-k^2\frac{\hbar}{4m\omega}} \quad (4)$$

the last equality come from only the first terms in the expansion of the exponentials (which is 1) survive.

Problem 2

From a previous problem set in the S_z basis the angular momentum operators are

$$S_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad S_y = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad S_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \quad (5)$$

The unperturbed Hamiltonian is

$$H^{(0)} = AS_z^2 \quad (6)$$

and its eigenstates are the eigenstates of S_z^2 and S_z . In the S_z basis they can be written as

$$\psi_{-1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \psi_0 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \psi_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad (7)$$

with eigenvalues $A\hbar^2/2$, 0 and $A\hbar^2/2$ respectively, so energy level $A\hbar^2/2$ is degenerate. We use the eigenvectors of the

$$H^{(1)} = B(S_x^2 - S_y^2) \quad (8)$$

perturbation to lift the degeneracy. They are

$$\psi_- = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \quad \psi_0 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \psi_+ = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad (9)$$

with energies $-B\hbar^2$, 0 and $B\hbar^2$ respectively, so the states suitable for perturbation theory are ψ_- , ψ_0 and ψ_+ and the energies are

$$A\hbar^2 - B\hbar^2, \quad 0, \quad A\hbar^2 + B\hbar^2, \quad (10)$$

respectively.

Problem 3

The additional term $H_1 = \gamma|\frac{p}{m}|$ corresponds to the dissipation. It should lower the mechanical energy of the particle when it moves. Therefore, γ should be negative.

As the hint suggested, the operator is less ambiguous if we work in momentum space. The ground state wave function in position space is $\psi(x) = A' \exp[-m\omega x^2/2\hbar]$. Fourier transforming it into momentum space yields

$$\psi(p) = A e^{-\frac{p^2}{2m\hbar\omega}} \quad (11)$$

where $A = (\pi m \hbar \omega)^{-1/4}$ is the normalization constant. The shift in ground state energy then follows

$$\begin{aligned}\Delta E &= \langle 0 | H_1 | 0 \rangle \\ &= \int_{-\infty}^{\infty} dp \psi^2(p) \gamma \left| \frac{P}{m} \right| \\ &= 2 \frac{\gamma}{m} A^2 \int_{-\infty}^{\infty} dp p e^{-\frac{p^2}{2m\hbar\omega}} \\ &= \gamma \sqrt{\frac{\hbar\omega}{m\pi}}\end{aligned}\tag{12}$$