Physics 125b Midterm Solutions

February 28, 2017

Problem 1

a)as of representations of $SO(3)$ group, we have

$$1 \otimes 1 \otimes \frac{1}{2} = (2 \oplus 1 \oplus 0) \otimes \frac{1}{2} = \frac{5}{2} \oplus \frac{3}{2} \oplus \frac{3}{2} \oplus \frac{1}{2} \oplus \frac{1}{2} \tag{1}$$

one can check we have total dimension 18 on both side

b)we notice from the above that $2 \otimes \frac{1}{2} \to \frac{5}{2}$ thus we have

$$\begin{aligned} |5/2, m\rangle &= \sum_{m_1, m_2} \langle 2, m_1; 1/2, m_2 | 5/2, m\rangle (|2, m_1\rangle \otimes |1/2, m_2\rangle) \\ &= \sum_{m_1, m_2, m_3, m_4} \langle 2, m_1; 1/2, m_2 | 5/2, m\rangle \langle 1, m_3; 1, m_4 | 2, m_1\rangle (|1, m_3\rangle \otimes |1, m_4\rangle \otimes |1/2, m_2\rangle) \end{aligned} \tag{2}$$

the sum over m_i gives only non-zero contribution for those satifies $m_3 + m_4 = m_1, m_1 + m_2 = m$, which should be implicit in the definition of CG coefficients. one can also use $J_i^{\pm}$ to directly compute the coefficients from the highest state $|5/2, 5/2\rangle = |1, 1\rangle \otimes |1, 1\rangle \otimes |1/2, 1/2\rangle$

Problem 2

From the case considered in the problem, we have

$$H_0 = \frac{p^2}{2m} + \frac{m\omega^2 r^2}{2} \tag{3}$$

$$H_1 = -\frac{m\omega^2 r^2}{2}(r > r_0) \tag{4}$$

we can compute first order perturbation of ground state energy directly by

$$E_{gs}^{(1)} = \langle \phi_0 | H_1 | \phi_0 \rangle \tag{5}$$

with given $|\phi_0\rangle = (\frac{m\omega}{\pi\hbar})^{3/4} \exp(-\frac{m\omega r^2}{2\hbar})$ by doing integral we get

$$E_{gs}^{(1)} = -(\frac{m\omega}{\pi\hbar})^{3/2}(4\pi) \int_{r_0}^{\infty} \exp(-\frac{m\omega r^2}{\hbar}) \frac{m\omega^2 r^2}{2} r^2 dr = -\frac{2\omega\hbar}{\sqrt{\pi}} J(\sqrt{\frac{m\omega}{\hbar}} r_0, 4) \tag{6}$$

which should be a small perturbation as $E_{gs}^{(1)}$ is small compare to the energy scale of the system $\hbar\omega$, which means $J(\sqrt{\frac{m\omega}{\hbar}} r_0, 4) \ll 1$ (consider property of this function) and $\sqrt{\frac{m\omega}{\hbar}} r_0 \gg 1$

Problem 3

The unperturbed Hamiltonian is $H_0 = \vec{p}^2/(2m) - e^2/r$ with perturbation $H_1 = -2\epsilon \sin\omega t \, e^2/r$. To first order in perturbation theory, the transition coefficient on an $n = 2$ excited state with angular momentum (l, m) is

$$\begin{aligned} d_f &= -\frac{i}{\hbar} \int_0^{\Delta t} \langle 2, l, m | H_1 | 1, 0, 0 \rangle e^{i(E_2 - E_1)t'} \, dt' \\ &= \frac{2i\epsilon e^2}{\hbar} \langle 2, l, m | \frac{1}{r} | 1, 0, 0 \rangle \int_0^{\Delta t} \sin\omega t' e^{i(E_2 - E_1)t'} \, dt' \\ &= \frac{2i\epsilon e^2}{\hbar} \langle 2, l, m | \frac{1}{r} | 1, 0, 0 \rangle \int_0^{\Delta t} \omega t' \, dt' \\ &= \frac{i\epsilon e^2}{\hbar} \omega (\Delta t)^2 \langle 2, l, m | \frac{1}{r} | 1, 0, 0 \rangle \end{aligned} \tag{7}$$

where we keep the leading order in small t' regime in the third line. The perturbation is spherical symmetric, $[H_1, L^2] = [H_1, L_z] = 0$, so it preserves the angular momentum quantum numbers. Therefore, the transition matrix element $\langle 2, l, m | \frac{1}{r} | 1, 0, 0 \rangle$ vanishes except for $(l, m) = (0, 0)$. Plugging the Coulumb wave functions $\psi_{1,0,0} = e^{-r/a_0}/\sqrt{\pi a_0^3}$ and $\psi_{2,0,0} = (2 - r/a_0)e^{-r/(2a_0)}/\sqrt{32\pi a_0^3}$ into the matrix element yields

$$\begin{aligned} \langle 2, 0, 0 | \frac{1}{r} | 1, 0, 0 \rangle &= \int_0^{\infty} dr' 4\pi r' \psi_{2,0,0}(r') \psi_{1,0,0}(r') \\ &= \frac{4\sqrt{2}}{27 a_0}. \end{aligned} \tag{8}$$

Then, we find

$$d_{2,0,0} = \frac{i\epsilon e^2 4\sqrt{2}}{27\hbar a_0}\omega(\Delta t)^2 = i\epsilon\frac{4\sqrt{2}\alpha c}{27a_0}\omega(\Delta t)^2 \tag{9}$$

which is translated into the probability as

$$P_{2,0,0} = |d_{2,0,0}|^2 = \epsilon^2\frac{32\alpha^2}{729a_0^2}c^2\omega^2\Delta t^4. \tag{10}$$

Here α is the dimensionless structure constant. We can see that the probability is proportional to $(c\Delta t/a_0)^2(\omega\Delta t)^2$ which is dimensionless.

Note that the above analysis only works at first order. There might be higher order correction for other excited states, $(l,m) = (1,\pm 1)$ or $(l,m) = (1,0)$. However, we notice that $[H_1, L^2] = [H_1, L_z] = 0$. This guarantees that the time dependent state

$$|\psi(t')\rangle = e^{-i(H_0+H_1)t'/\hbar}|1,0,0\rangle \tag{11}$$

cannot have any component on $(l,m) \neq (0,0)$, *i.e.*, $\langle n,l,m|\psi(t')\rangle = 0$ for $l \neq 0$ or $m \neq 0$. The mathematical proof can be found in the discussion of the selection rule in Shankar. Thus, the transition probability is zero to all orders for other $n = 2$ excited states.

Grading Scheme:

- 3pt: deriving the matrix elelemnt $\langle 2,0,0|\frac{1}{r}|1,0,0\rangle = \frac{4\sqrt{2}}{27a_0}$, (-1) point if the final result is wrong.
- 2pt: time dependent integral $\int_0^{\Delta t}\sin\omega t' e^{i(E_2-E_1)t'}\,dt'$, (-1) point if it wans't simplified at leading order $\omega\Delta t^2$.
- 1pt: check dimension of the final probability
- 2pt: show 1st order matrix elelemnt $\langle 2,l,m|\frac{1}{r}|1,0,0\rangle = 0$ for $(l,m) \neq (0,0)$.
- 2pt: argue that the transition is zero to all orders. This is an challenging part. Thus, I give full credit even if the student only mentioned that $[H_1, L^2] = [H_1, L_z] = 0$ but didn't fully argue that $\langle n,l,m|\psi(t')\rangle = 0$ to all order. Students who only showed 1st order matrix elelemnt is zero from spherical harmonics orthogonality would lose two points here.