Physics 125b Midterm Solutions

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Problem 1

From the case considered in the problem, we have

$$H_0 = \frac{p^2}{2m} + \frac{m\omega^2 r^2}{2} \tag{1}$$

$$H_1 = -\frac{m\omega^2 r^2}{2}(r > r_0) \tag{2}$$

we can compute first order perturbation of ground state energy directly by

$$E_{gs}^{(1)} = \langle\phi_0|H_1|\phi_0\rangle \tag{3}$$

with given $|\phi_0\rangle = (\frac{m\omega}{\pi\hbar})^{3/4}\exp(-\frac{m\omega r^2}{2\hbar})$ by doing integral we get

$$E_{gs}^{(1)} = -(\frac{m\omega}{\pi\hbar})^{3/2}(4\pi)\int_{r_0}^{\infty}\exp(-\frac{m\omega r^2}{\hbar})\frac{m\omega^2 r^2}{2}r^2 dr = -\frac{2\omega\hbar}{\sqrt{\pi}}J(\sqrt{\frac{m\omega}{\hbar}}r_0, 4) \tag{4}$$

which should be a small perturbation as $E_{gs}^{(1)}$ is small compare to the energy scale of the system $\hbar\omega$, which means $J(\sqrt{\frac{m\omega}{\hbar}}r_0, 4) \ll 1$ (consider property of this function) and $\sqrt{\frac{m\omega}{\hbar}}r_0 \gg 1$

Problem 2

(a)

We start with ground state $|\phi\rangle = |-1/2\rangle$ (here we spesify the state by its eigenvalue under S_z) with first order time dependent perturbation, we have

$$\langle 1/2|\phi_{final}\rangle \to -\frac{i}{\hbar}\int_0^T \langle 1/2|H^1(t)|-1/2\rangle \exp^{i\omega t} dt = -\frac{g}{2\omega}(\exp^{i\omega t} - 1) \quad (5)$$

where we have

$$\langle 1/2|H^1(t)|-1/2\rangle = \frac{g\hbar}{2} \quad (6)$$

$$\omega = \frac{E^0_{1/2} - E^0_{-1/2}}{\hbar} = \mu \quad (7)$$

for the transition rate at first order

$$P = |\langle 1/2|\phi_{final}\rangle|^2 = \frac{g^2}{2\mu^2}(1 - \cos\mu T) \quad (8)$$

(b)

We notice that first order perturbation for the ground state energy vanishes, so we use second order to get nn-trivial contribution

$$E^{(2)}_{gs} = \frac{|\langle 1/2|H^1(t)|-1/2\rangle|^2}{E^{(0)}_{-1/2} - E^{(0)}_{1/2}} = -\frac{g^2\hbar}{4\mu} \quad (9)$$

for exact ground energy after the perturbation, we calculate the lowest eigenvalue for the perturbed Hamiltonian $H = H_0 + H_1$

$$E_{gs} = -\frac{\hbar}{2}\sqrt{\mu^2 + g^2} \to -\frac{\hbar\mu}{2} - \frac{g^2\hbar}{4\mu} + O(g^4) \quad (10)$$

Problem 3

The unperturbed Hamiltonian is $H_0 = \vec{p}^2/(2m) - e^2/r$ with perturbation $H_1 = -2\epsilon \sin\omega t\, e^2/r$. To first order in perturbation theory, the transition coefficient on an $n = 2$ excited state with angular momentum (l, m) is

$$\begin{aligned} d_f &= -\frac{i}{\hbar}\int_0^{\Delta t} \langle 2,l,m|H_1|1,0,0\rangle e^{i(E_2-E_1)t'} dt' \\ &= \frac{2i\epsilon e^2}{\hbar}\langle 2,l,m|\frac{1}{r}|1,0,0\rangle \int_0^{\Delta t} \sin\omega t' e^{i(E_2-E_1)t'} dt' \\ &= \frac{2i\epsilon e^2}{\hbar}\langle 2,l,m|\frac{1}{r}|1,0,0\rangle \int_0^{\Delta t} \omega t'\, dt' \\ &= \frac{i\epsilon e^2}{\hbar}\omega(\Delta t)^2\langle 2,l,m|\frac{1}{r}|1,0,0\rangle \end{aligned} \quad (11)$$

where we keep the leading order in small t' regime in the third line. The perturbation is spherical symmetric, $[H_1, L^2] = [H_1, L_z] = 0$, so it preserves the angular momentum quantum numbers. Therefore, the transition matrix element $\langle 2, l, m|\frac{1}{r}|1, 0, 0\rangle$ vanishes except for $(l, m) = (0, 0)$. Plugging the Coulumb wave functions $\psi_{1,0,0} = e^{-r/a_0}/\sqrt{\pi a_0^3}$ and $\psi_{2,0,0} = (2 - r/a_0)e^{-r/(2a_0)}/\sqrt{32\pi a_0^3}$ into the matrix element yields

$$\begin{aligned}\langle 2, 0, 0|\frac{1}{r}|1, 0, 0\rangle &= \int_0^\infty dr' 4\pi r' \psi_{2,0,0}(r')\psi_{1,0,0}(r') \\ &= \frac{4\sqrt{2}}{27a_0}.\end{aligned} \tag{12}$$

Then, we find

$$d_{2,0,0} = \frac{i\epsilon e^2 4\sqrt{2}}{27\hbar a_0}\omega(\Delta t)^2 = i\epsilon\frac{4\sqrt{2}\alpha c}{27a_0}\omega(\Delta t)^2 \tag{13}$$

which is translated into the probability as

$$P_{2,0,0} = |d_{2,0,0}|^2 = \epsilon^2\frac{32\alpha^2}{729a_0^2}c^2\omega^2\Delta t^4. \tag{14}$$

Here α is the dimensionless structure constant. We can see that the probability is proportional to $(c\Delta t/a_0)^2(\omega\Delta t)^2$ which is dimensionless.

Note that the above analysis only works at first order. There might be higher order correction for other excited states, $(l, m) = (1, \pm 1)$ or $(l, m) = (1, 0)$. However, we notice that $[H_1, L^2] = [H_1, L_z] = 0$. This guarantees that the time dependent state

$$|\psi(t')\rangle = e^{-i(H_0+H_1)t'/\hbar}|1, 0, 0\rangle \tag{15}$$

cannot have any component on $(l, m) \neq (0, 0)$, *i.e.*, $\langle n, l, m|\psi(t')\rangle = 0$ for $l \neq 0$ or $m \neq 0$. The mathematical proof can be found in the discussion of the selection rule in Shankar. Thus, the transition probability is zero to all orders for other $n = 2$ excited states.