Physics 125b – Final Solutions

Problem 1

(a)

We may simply assume μ is positive, then we start with ground state $|\phi\rangle = |-1/2\rangle$ (here we spesify the state by its eigenvalue under S_z) with first order time dependent perturbation, we have

$$\langle 1/2|\phi_{final}\rangle \to -\frac{i}{\hbar}\int_0^T \langle 1/2|H^1(t)|-1/2\rangle \exp^{i\omega t} dt = -\frac{g}{2\omega}(\exp^{i\omega t} - 1) \quad (1)$$

where we have

$$\langle 1/2|H^1(t)|-1/2\rangle = \frac{g\hbar}{2}$$
$$\omega = \frac{E^0_{1/2} - E^0_{-1/2}}{\hbar} = \mu$$

for the transition rate at first order

$$P = |\langle 1/2|\phi_{final}\rangle|^2 = \frac{g^2}{2\mu^2}(1 - \cos\mu T) \quad (2)$$

(b)

We notice that first order perturbation for the ground state energy vanishes, so we use second order to get nn-trivial contribution

$$E^{(2)}_{gs} = \frac{|\langle 1/2|H^1(t)|-1/2\rangle|^2}{E^{(0)}_{-1/2} - E^{(0)}_{1/2}} = -\frac{g^2\hbar}{4\mu} \quad (3)$$

for exact ground energy after the perturbation, we calculate the lowest eigenvalue for the perturbed Hamiltonian $H = H_0 + H_1$

$$E_{gs} = -\frac{\hbar}{2}\sqrt{\mu^2 + g^2} \to -\frac{\hbar\mu}{2} - \frac{g^2\hbar}{4\mu} + O(g^4) \quad (4)$$

Problem 2

We compute $f(\theta, \phi)$,

$$f(\theta,\phi) = -\frac{m}{2\pi\hbar^2}\int e^{-i\vec{q}\cdot\vec{r}} V(\vec{r}) d^3\vec{r}. \quad (5)$$

The momentum is

$$\vec{q} = \vec{k}_f - \vec{k}_i = (k(\cos\theta - 1), k\sin\theta\cos\phi, k\sin\theta\sin\phi), \quad (6)$$

so that

$$\begin{aligned} f(\theta,\phi) &= \frac{m}{2\pi\hbar^2}\int_{x,y,z=-L}^{L} e^{-i[k\sin\theta\cos\phi\, x + k\sin\theta\sin\phi\, y + k(\cos\theta-1)z]} V_0 dxdydz \\ &= \frac{4m}{\pi\hbar^2}\frac{\sin[kL\sin(\theta)\sin(\phi)]\sin[kL(\cos(\theta)-1)]\sin[kL\sin(\theta)\cos(\phi)]}{k^3(\cos(\theta)-1)\sin\phi\cos\phi\sin^2\theta}. \end{aligned}$$

The differential cross-section is the absolute value of $f(\theta, \phi)$ squared.

Problem 3

μ has units of energy $\times$ length$^{3/2}$, which is the same as length$^{1/2}$. We have

$$\langle \vec{k}_2, \vec{k}_3 | H_{\text{int}} | \vec{k}_1 \rangle = \mu \langle 0 | a(\vec{k}_3) a(\vec{k}_2) \int d^3x \left(\phi^\dagger(\vec{x}) \phi^\dagger(\vec{x}) \phi(\vec{x}) + \phi(\vec{x}) \phi(\vec{x}) \phi^\dagger(\vec{x}) \right) a^\dagger(\vec{k}_1) | 0 \rangle \tag{7}$$

and

$$\phi(\vec{x}) = \int \frac{d^3\vec{k}}{(2\pi)^{3/2}} e^{i\vec{k}\cdot\vec{x}} a(\vec{k}). \tag{8}$$

Plugging this in gives two terms that look like

$$aaa^\dagger a^\dagger a a^\dagger \qquad \text{and} \qquad aaaaa^\dagger a^\dagger. \tag{9}$$

The second term has too many a's and vanishes and (with the normalization from class) we have

$$\begin{aligned}
\langle \vec{k}_2, \vec{k}_3 | H_{\text{int}} | \vec{k}_1 \rangle &= \mu \langle 0 | \int d^3x \frac{d^3\vec{l}_1 d^3\vec{l}_2 d^3\vec{l}_3}{(2\pi)^{9/2}} e^{i(\vec{l}_1 + \vec{l}_2 - \vec{l}_3)\cdot\vec{x}} a(\vec{k}_3) a(\vec{k}_2) a^\dagger(\vec{l}_1) a^\dagger(\vec{l}_2) a(\vec{l}_3) a^\dagger(\vec{k}_1) | 0 \rangle \\
&= \mu \langle 0 | \int d^3x \delta^{(3)}(\vec{k}_1 - \vec{l}_3) e^{i(\vec{l}_1 + \vec{l}_2 - \vec{l}_3)\cdot\vec{x}} \frac{d^3\vec{l}_1 d^3\vec{l}_2 d^3\vec{l}_3}{(2\pi)^{9/2}} a(\vec{k}_3) a(\vec{k}_2) a^\dagger(\vec{l}_1) a^\dagger(\vec{l}_2) | 0 \rangle \\
&= \mu \langle 0 | \int d^3x e^{i(\vec{l}_1 + \vec{l}_2 - \vec{k}_1)\cdot\vec{x}} \frac{d^3\vec{l}_1 d^3\vec{l}_2}{(2\pi)^{9/2}} a(\vec{k}_3) a(\vec{k}_2) a^\dagger(\vec{l}_1) a^\dagger(\vec{l}_2) | 0 \rangle \\
&= \mu \langle 0 | \int d^3x e^{i(\vec{l}_1 + \vec{l}_2 - \vec{k}_1)\cdot\vec{x}} \frac{d^3\vec{l}_1 d^3\vec{l}_2}{(2\pi)^{9/2}} a(\vec{k}_3) \left(a^\dagger(\vec{l}_1) a(\vec{k}_2) + \delta^{(3)}(\vec{l}_1 - \vec{k}_2) \right) a^\dagger(\vec{l}_2) | 0 \rangle \\
&= \mu \langle 0 | \int d^3x e^{i(\vec{l}_1 + \vec{l}_2 - \vec{k}_1)\cdot\vec{x}} \frac{d^3\vec{l}_1 d^3\vec{l}_2}{(2\pi)^{9/2}} \left(\delta^{(3)}(\vec{l}_1 - \vec{k}_3) \delta^{(3)}(\vec{k}_2 - \vec{l}_2) + \delta^{(3)}(\vec{l}_1 - \vec{k}_2) \delta^{(3)}(\vec{l}_2 - \vec{k}_3) \right) | 0 \rangle \\
&= 2\mu \int \frac{d^3x}{(2\pi)^{9/2}} e^{i(\vec{k}_2 + \vec{k}_3 - \vec{k}_1)\cdot\vec{x}} \\
&= \frac{2\mu}{(2\pi)^{3/2}} \delta^{(3)}(\vec{k}_2 + \vec{k}_3 - \vec{k}_1).
\end{aligned}$$

The δ-function is imposing momentum conservation.