Ph125a Homework 2 Solutions

Fall 2016

Problem 1

$|K^0\rangle$ and $|\bar{K}^0\rangle$ are orthogonal states with unit norm, representing the particles K^0 and $\bar{K}^0$ (these particles are called neutral kaons). We approximate the dynamics of the two kaons as a two-level system with an effective Hamiltonian that is not Hermitian, where probabilities are not conserved as the kaons can decay. The Hamiltonian is $H = M - \frac{i}{2}\Gamma$, where the mass and decay terms are Hermitian.

a) Suppose that the off-diagonal components M_{12} and Γ_{12} are zero (meaning that the kaons cannot interact). We write the Hamiltonian in the $|K^0\rangle$ and $|\bar{K}^0\rangle$ basis as

$$H = \begin{pmatrix} M - \frac{i}{2}\Gamma & 0 \\ 0 & M - \frac{i}{2}\Gamma \end{pmatrix} \quad \text{with} \quad |K^0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |\bar{K}^0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} .$$

The states $|K^0\rangle$ and $|\bar{K}^0\rangle$, are eigenstates of this diagonal Hamiltonian. Recall that we time-evolve by acting with the operator $U(t) = e^{-itH/\hbar}$. If we start with the system in the state $|K^0\rangle$, then

$$|\psi(t)\rangle = e^{-itH/\hbar} |K^0\rangle = e^{-it(M-\frac{i}{2}\Gamma)/\hbar} |K^0\rangle \ .$$

as $|K^0\rangle$ is an energy eigenstate. Note that the time evolved state does not have unit norm

$$\big|\langle\psi(t)|\psi(t)\rangle\big| = e^{-it(M-\frac{i}{2}\Gamma)/\hbar + it(M+\frac{i}{2}\Gamma)/\hbar} \langle K^0|K^0\rangle = e^{-\Gamma t/\hbar} \, .$$

Usually, the time-evolving energy eigenstates by a normal Hamiltonian stays in the eigenstate, i.e. we evolve by a phase. Equivalently, when the Hamiltonian is Hermitian, then time evolution is a unitary operator, but here this is no longer true. We are time-evolving with a non-unitary operator and the probabilities are no longer conserved; the particle can decay!

To correspond to decay, Γ should be positive, meaning the overlap of states becomes smaller at later times. If we start out in $|K^0\rangle$, then the probability we remain in that state at a later time is

$$\big|\langle K^0|\psi(t)\rangle\big|^2 = e^{-\Gamma t/\hbar} \, ,$$

and if we have N kaons at $t = 0$, then we will have $Ne^{-\Gamma T/\hbar}$ at time T. The decay time of neutral kaons K^0 and $\bar{K}^0$, in the real world is $\hbar/\Gamma \sim 10^{-10}$ sec. This may seem very short, but is fairly long-lived as far as particle physics is concerned. For comparison, the excited states of a photon live for about $\sim 10^{-23}$ sec before decaying. In hadronic physics, neutral kaons are considered 'stable' particles.

b) Now we consider nonzero real off-diagonal terms in the effective Hamiltonian

$$H = \begin{pmatrix} M - \frac{i}{2}\Gamma & M_{12} - \frac{i}{2}\Gamma_{12} \\ M_{12} - \frac{i}{2}\Gamma_{12} & M - \frac{i}{2}\Gamma \end{pmatrix} \, ,$$

again written in the $|K^0\rangle$ and $|\bar{K}^0\rangle$ basis. The off-diagonal terms must be equal as the mass and decay matricies are Hermitian, $M_{12} = M_{21}$ and $\Gamma_{12} = \Gamma_{21}$. Noting that the Hamiltonian is nice and symmetric, we solve for the eigenvalues and eigenvectors

$$H = \begin{pmatrix} a & b \\ b & a \end{pmatrix} \quad \to \quad \lambda_\pm = a \pm b \qquad \text{with eigenstates } |\psi_\pm\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \pm 1 \end{pmatrix},$$

where the energy eigenvalues are

$$\lambda_\pm = M - \frac{i}{2}\Gamma \pm \left(M_{12} - \frac{i}{2}\Gamma_{12}\right).$$

We should not be surprised that the eigenstates are that of the σ^x Pauli matrix, given that our Hamiltonian can be written as $H = a\mathbb{1} + b\sigma^x$.

c) We want to find the time-evolution of the state $|K^0\rangle$. As we found in part b, the eigenstates of the Hamiltonian with off-diagonal terms are no longer the $|K^0\rangle$ and $|\bar{K}^0\rangle$ states. We could solve the Schrödinger equation, $H|\psi\rangle = i\hbar\frac{\partial}{\partial t}|\psi\rangle$, but let's instead recall that the general solution to the time-dependent Schrödinger equation can be written as a linear combination of energy eigenstates

$$|\psi(t)\rangle = \sum_n c_n e^{-iE_n t/\hbar} |\psi_n\rangle \; .$$

Writing the energy eigenstates of H, $|\psi_\pm\rangle$, as a sum of $|K^0\rangle$ and $|\bar{K}^0\rangle$

$$|\psi_+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\big(|K^0\rangle + |\bar{K}^0\rangle\big), \quad \text{and} \quad |\psi_-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}}\big(|K^0\rangle - |\bar{K}^0\rangle\big).$$

We can then write a general time-evolved state as

$$|\psi(t)\rangle = \frac{c_1}{\sqrt{2}}e^{-i\lambda_+ t/\hbar}\big(|K^0\rangle + |\bar{K}^0\rangle\big) + \frac{c_2}{\sqrt{2}}e^{-i\lambda_- t/\hbar}\big(|K^0\rangle - |\bar{K}^0\rangle\big).$$

Given the initial condition that we start out in $|K^0\rangle$ at $t = 0$, we have $c_1 = c_2 = \frac{1}{\sqrt{2}}$. At time t, the system is in the state

$$|\psi(t)\rangle = \frac{1}{2}\left(e^{-i\lambda_+ t/\hbar} + e^{-i\lambda_- t/\hbar}\right)|K^0\rangle + \frac{1}{2}\left(e^{-i\lambda_+ t/\hbar} - e^{-i\lambda_- t/\hbar}\right)|\bar{K}^0\rangle \; .$$

The probability that we find the system in the state $\bar{K}^0$ is then

$$\text{Prob}(K^0 \to \bar{K}^0) = \left|\langle \bar{K}^0|\psi(t)\rangle\right|^2 = \frac{1}{4}\left|e^{-i\lambda_+ t/\hbar} - e^{-i\lambda_- t/\hbar}\right|^2 = \frac{1}{4}e^{\Gamma t/\hbar}\left|e^{(-iM_{12} - \frac{1}{2}\Gamma_{12})t/\hbar} - e^{(iM_{12} + \frac{1}{2}\Gamma_{12})t/\hbar}\right|^2$$

$$= \frac{1}{4}e^{-\Gamma t/\hbar}\left(e^{\Gamma_{12}t/\hbar} + e^{-\Gamma_{12}t/\hbar} - 2\cos(2M_{12}t/\hbar)\right).$$

As a sanity check, we can take the off-diagonal components to zero and we find the probability goes to zero, which is the expectation from part a: although the kaons can decay if they are energy eigenstates (like they were in part a) they should not mix.

Problem 2

We have three compatible observables Ω, Λ, Γ, meaning they are simultaneously diagonalizable and share a complete set of eigenvectors, labeled by eigenvalues of the operators as $|\omega_i, \lambda_i, \gamma_i\rangle$. We are asked to consider the state

$$|\psi\rangle = \frac{1}{\sqrt{3}}|\omega_1, \lambda_1, \gamma_3\rangle - \frac{1}{\sqrt{6}}|\omega_1, \lambda_2, \gamma_4\rangle + \frac{i}{\sqrt{2}}|\omega_2, \lambda_1, \gamma_4\rangle$$

a) The state has unit norm

$$\sqrt{\langle\psi|\psi\rangle} = \sqrt{\frac{1}{3} + \frac{1}{6} + \frac{1}{2}} = 1\,.$$

Recall that projective measurement of an operator in QM, given the spectral decomposition of an operator into an orthonormal basis of eigenvectors $\mathcal{O} = \sum_\alpha \alpha P_\alpha = \sum_\alpha \alpha|\alpha\rangle\langle\alpha|$, where $P_\alpha = |\alpha\rangle\langle\alpha|$ is the projector onto the eigenstate $|\alpha\rangle$, measuring in a state $|\psi\rangle$ means that the probability to measure α is given by

$$p(\alpha) = \langle\psi|P_\alpha|\psi\rangle\ ,$$

and given the measurement outcome α, after the measurement, the system is in the state

$$|\psi_\alpha\rangle = \frac{P_\alpha|\psi\rangle}{\sqrt{\langle\psi|P_\alpha|\psi\rangle}}\,.$$

Suppose, in the state $|\psi\rangle$, we measure the operators in the order Ω, then Λ, and then Γ. The possible measurement outcomes with the associated probabilities are

$$|\psi\rangle \xrightarrow{\Omega} p(\omega_1) = \tfrac{1}{2},\ \sqrt{\tfrac{2}{3}}|\omega_1, \lambda_1, \gamma_3\rangle - \tfrac{1}{\sqrt{3}}|\omega_1, \lambda_2, \gamma_4\rangle \begin{cases} \xrightarrow{\Lambda} p(\lambda_1) = \tfrac{2}{3},\ |\omega_1, \lambda_1, \gamma_3\rangle \xrightarrow{\Gamma} p(\gamma_3) = 1,\ |\omega_1, \lambda_1, \gamma_3\rangle \\ \xrightarrow{\Lambda} p(\lambda_2) = \tfrac{1}{3},\ |\omega_1, \lambda_2, \gamma_4\rangle \xrightarrow{\Gamma} p(\gamma_4) = 1,\ |\omega_1, \lambda_2, \gamma_4\rangle \end{cases}$$

$$|\psi\rangle \xrightarrow{\Omega} p(\omega_2) = \tfrac{1}{2},\ |\omega_2, \lambda_1, \gamma_4\rangle \xrightarrow{\Lambda} p(\lambda_1) = 1,\ |\omega_2, \lambda_1, \gamma_4\rangle \xrightarrow{\Gamma} p(\gamma_4) = 1,\ |\omega_2, \lambda_1, \gamma_4\rangle$$

Therefore, the probability of measuring ω_1, λ_2, and γ_4 is 1/6.

b) We know that as the operators are simultaneously diagonalizable, they must commute and therefore the order we measure them in does not matter. Let's illustrate one other measurement order, consider measuring Γ, Ω, Λ

$$|\psi\rangle \xrightarrow{\Gamma} p(\gamma_3) = \tfrac{1}{3},\ |\omega_1, \lambda_1, \gamma_3\rangle \xrightarrow{\Omega} p(\omega_1) = 1,\ |\omega_1, \lambda_1, \gamma_3\rangle \xrightarrow{\Lambda} p(\lambda_1) = 1,\ |\omega_1, \lambda_1, \gamma_3\rangle$$

$$|\psi\rangle \xrightarrow{\Gamma} p(\gamma_4) = \tfrac{2}{3},\ \text{-}\tfrac{1}{2}|\omega_1, \lambda_2, \gamma_4\rangle + \tfrac{i\sqrt{3}}{2}|\omega_2, \lambda_1, \gamma_4\rangle \begin{cases} \xrightarrow{\Omega} p(\omega_1) = \tfrac{1}{4},\ |\omega_1, \lambda_2, \gamma_4\rangle \xrightarrow{\Lambda} p(\lambda_2) = 1,\ |\omega_1, \lambda_2, \gamma_4\rangle \\ \xrightarrow{\Omega} p(\omega_2) = \tfrac{3}{4},\ |\omega_2, \lambda_1, \gamma_4\rangle \xrightarrow{\Lambda} p(\lambda_1) = 1,\ |\omega_2, \lambda_1, \gamma_4\rangle \end{cases}$$

In any of the six measurement orders, we can measure $|\omega_1, \lambda_1, \gamma_3\rangle$ with probability 1/3, $|\omega_1, \lambda_2, \gamma_4\rangle$ with probability 1/6, or $|\omega_2, \lambda_1, \gamma_4\rangle$ with probability 1/2.

Problem 3

a) Given the linear operators A, B and C,

$$[A, BC] = ABC + BCA = ABC - BCA - BAC + BAC = [A, B]C + B[A, C] .$$

and

$$\begin{aligned}[A, [B, C]] + [B, [C, A]] &+ [C, [A, B]] \\ &= [A, B]C + B[A, C] - [A, C]B - C[A, B] + [B, C]A + C[B, A] \\ &\quad - [B, A]C - A[B, C] + [C, A]B + A[C, B] - [C, B]A - B[C, A] \\ &= 0 .\end{aligned}$$

This cyclic identity is called the Jacobi relation.

b) We see that

$$\begin{aligned}[A, B^n] &= [A, B^{n-1}]B + B^{n-1}[A, B] = [A, B^{n-1}]B + cB^{n-1} \\ &= \big([A, B^{n-2}]B + B^{n-2}[A, B]\big)B + cB^{n-1} = [A, B^{n-2}]B^2 + 2cB^{n-1} ,\end{aligned}$$

proceeding by induction

$$[A, B^n] = [A, B^{n-(n-1)}]B^{n-1} + (n-1)cB^{n-1} = ncB^{n-1} .$$

Problem 4

c) Using the canonical commutation relation $[X, P] = i\hbar$, we evaluate the commutator $[X, e^{iPa}]$ as

$$[X, e^{iPa}] = [X, 1 + (iPa) + \ldots] = \sum_{n=0}^{\infty} \frac{(ia)^n}{n!}[X, P^n] = \sum_{n=1}^{\infty} \frac{(ia)^n}{(n-1)!} i\hbar P^{n-1} ,$$

where the $n = 0$ term in the sum was zero. We can shift the sum by defining $n' = n - 1$

$$[X, e^{iPa}] = -a\hbar \sum_{n'=0}^{\infty} \frac{(ia)^{n'}}{n'!} P^{n'} = -a\hbar e^{iPa} .$$

d) Using the above result, we want to show that $e^{iPa} |x\rangle$ is an eigenstate of the position operator X, where we have the awesome equation $X |x\rangle = x |x\rangle$,

$$X\left(e^{iPa} |x\rangle\right) = \left([X, e^{iPa}] + e^{iPa}X\right) |x\rangle = (x - a\hbar)e^{iPa} |x\rangle ,$$

and is thus an eigenstate of X with eigenvalue $x - a\hbar$.