Ph125 Review Session

Plan:
- Axioms of QM
- Angular Momentum
- Density Matrices and mixed states

Basic Facts about QM

There are four main axioms of QM, these define: <u>states</u>, <u>observables</u>, <u>time evolution</u>, <u>measurement</u>
these are answers to the following questions

1) How do we represent our knowledge of a physical system? → <u>states</u>
2) what are the physical properties of the system we can measure? → <u>observables</u>
3) Given a state of our system, what will the system look like later? → <u>time evolution</u>
4) What happens when we measure (perturb/affect/look at) a QM system? → <u>measurement</u>

1) the state of a system is a complete description of it, all the data we could specify
   in CM: this would be the positions + momenta of a bunch of particles
   in QM: a state is a vector in a Hilbert space (vector space over $\mathbb{C}$)

$$|\psi\rangle \in \mathcal{H}$$

→ $\mathcal{H}$ has an inner product, i.e. given two vectors $|\psi\rangle, |\phi\rangle$ we can make a complex # out of them $\langle\phi|\psi\rangle = \#$

properties of $\mathcal{H}$: positivity $\||\psi\rangle\|^2 = \langle\psi|\psi\rangle \geq 0$ for $|\psi\rangle \neq 0$

linearity: $\langle\psi|\left(a|\phi_1\rangle + b|\phi_2\rangle\right) = a\langle\psi|\phi_1\rangle + b\langle\psi|\phi_2\rangle$

skew-sym: $\langle\phi|\psi\rangle = \langle\psi|\phi\rangle^{*}$

i.e. there is a dual space $\mathcal{H}^*$, $\langle\phi| \in \mathcal{H}^*$ in 1-to-1 correspondence (bijection) w/ $\mathcal{H}$

$\mathcal{H}$ is complete: the norm of a vector is determined by the inner product

$$\||\psi\rangle\| = \sqrt{\langle\psi|\psi\rangle}$$

$$\hookrightarrow \exists \; \mathbb{1} = \sum |\psi_i\rangle\langle\psi_i|$$

2) Observables

an observable is a Hermitian operator : (operators are linear map from $\mathcal{H} \to \mathcal{H}$, i.e. eats a vector and spits out another vector)

$A|a\rangle = |b\rangle$, $\langle a|A^+ = \langle b|$, Hermitian op $\underline{A = A^+}$

Hermitian ops have real eigenvalues and an ON eigenbasis

3) Time-evolution

the time evolution of a QM state is implemented by the Hamiltonian

$$H|\psi(t)\rangle = i\hbar \frac{d}{dt}|\psi(t)\rangle \quad \text{(schrödinger eq)}$$

$\left(|\psi(t)\rangle + \delta|\psi(\delta t)\rangle \right.$
$\quad d|\psi(t)\rangle = -i\,dt\,\frac{H}{\hbar}|\psi\rangle$

to first order $|\psi(t+\delta t)\rangle = \left(1 - iH\frac{\delta t}{\hbar}\right)|\psi(t)\rangle$

$\equiv U(\delta t)$ this is a unitary operator that moves the state forward by δt

apply this many times $\left(\lim_{N\to\infty}\left(1-\frac{x}{N}\right)^N = e^{-x}\right)$

$$|\psi(t)\rangle = U(t)|\psi(0)\rangle \quad \text{evolve a state to any time}$$

↳ manifestly solves Schrödinger eq.

(the Hamiltonian is the generator of time-translations)

note : energy eigenstates $H|\psi\rangle = E|\psi\rangle$ evolve by phases

$U|\psi\rangle = e^{-iEt/\hbar}|\psi\rangle$

4) Measurement

a) in QM, when we measure an observable A we get an eigenvalue of A

b) measurement changes the state, after measurement the system is in an eigenstate of A corresponding to the measured eigenvalue

a) $\text{Prob}(a;|\psi\rangle) = \langle\psi|\mathbb{P}_a|\psi\rangle$, if nondeg. $\Rightarrow$ $\mathbb{P}_a = |a\rangle\langle a|$ $\quad \text{Prob} = |\langle a|\psi\rangle|^2$

b) if we measure a

$$|\psi\rangle \xrightarrow[\text{get } a]{\text{measure } A} \frac{\mathbb{P}_a|\psi\rangle}{\sqrt{\langle\psi|\mathbb{P}_a|\psi\rangle}} \quad \text{if a nondegen} \to |a\rangle$$

You may worry that these axioms contradict each other
measurement is not linear, and is not unitary time evolution

⇒ Sidney Coleman's Quantum Mechanics in your face (Harvard lecture)

A few comments: Complete set of commuting operators, allows us to specify a ON basis of $\mathcal{H}$ by their eigenvalues

→ an operator w/ a completely non degenerate spectrum (no two eigenvalues are equal) is itself a complete set, i.e. X for particle on a line (but a bad one, not bounded)

→ more generally we find a # of operators that commute (i.e. H, J^2, J_z perhaps) and thus can be simultaneously diagonalized, the QM #s n, m, j uniquely label states

→ whats the point? in QM the choice of basis comes from observables
i.e. the things we put in $|\Psi\rangle$, $|n\ell m\rangle$, $|n a b c d \ldots\rangle$ are the values of the observables → eigenvalues of Herm. ops.

Axiom 5: we combine QM systems w/ tensor products

suppose we have two isolated spins (or qubits) $\mathcal{H}_1 = \text{span}\{|\uparrow\rangle_1, |\downarrow\rangle_1\}$
$\mathcal{H}_2 = \text{span}\{|\uparrow_2\rangle, |\downarrow_2\rangle\}$

the $\mathcal{H}$ of the total system is $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$

a basis of which is: $\{|\uparrow\rangle_1 \otimes |\uparrow\rangle_2, |\uparrow\rangle_1 \otimes |\downarrow\rangle_2, |\downarrow\rangle_1 \otimes |\uparrow\rangle_2, |\downarrow\rangle_1 \otimes |\downarrow\rangle_2\}$

i.e. $\mathcal{H} = \text{span}\{|\uparrow\uparrow\rangle, |\downarrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\downarrow\rangle\}$

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what is the wavefunction? Wavefunction = components

the data of the w.f. $\Psi(x)$ are the components of the vector in $\mathcal{H}$ in a particular basis

in the position basis: $\Psi(x) = \langle x|\Psi\rangle$ (i.e. $|\Psi\rangle = \int dx\, \Psi(x)\, |x\rangle$ w/ $\mathbb{1} = \int dx\, |x\rangle\langle x|$

the vector in $\mathcal{H}$ is a linear combination of vectors $|x\rangle$ w/ complex coefficients $\Psi(x)$

$$\rightarrow \langle x'|\Psi\rangle = \int dx\, \delta(x-x')\, \Psi(x) = \Psi(x')$$

and equivalently for the momentum basis: $\mathbb{1} = \int dp\, |p\rangle\langle p|$

$$|\psi\rangle = \int dp\, \tilde{\psi}(p)\, |p\rangle$$

how is $\tilde{\psi}(p)$ related to $\psi(x)$?

$$|\psi\rangle = \int dx\, \psi(x)\, \mathbb{1}\, |x\rangle = \int dx\, \psi(x) \int dp\, |p\rangle\langle p|x\rangle$$

$$= \int dp \left(\int \frac{dx}{\sqrt{2\pi\hbar}}\, e^{-ipx/\hbar}\, \psi(x) \right) |p\rangle$$

$$= \int dp\, \tilde{\psi}(p)\, |p\rangle$$

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consider overlap of basis states

$$\langle x|P|p\rangle = \langle x|P|p\rangle$$

$$-i\hbar\, \partial_x \langle x|p\rangle = p\, \langle x|p\rangle$$

soln

$$\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}}\, e^{-ipx/\hbar}$$

fix constants w/

$$\delta(x_1 - x_2) = \langle x_1|x_2\rangle = \int dp\, \langle x_1|p\rangle \langle p|x_2\rangle$$

$$= \int dp\, c^2\, e^{-ip(x_1 - x_2)/\hbar}$$

$$= 2\pi\hbar\, c^2\, \delta(x_1 - x_2)$$

Angular Momenta

we want to study QM in $d>1$ → physics is hard, we want to use symmetries of our system to simplify things

the operators corresponding to the symmetries are the ~~the~~ observables we want to consider and use to label states

Recall: if we have a cont. sym ⟷ conserved charge (QM Noether's theorem)

QM: consider symmetry is a U op acting on $\mathcal{H}$ which doesn't affect our system

$[U, H] = 0$ symmetry

applying a small U: $U = \mathbb{1} - i\epsilon Q + \dots$ and $[Q, H] = 0$

U is unitary → Q is hermitian, an observable

Hence this is a conservation law in that if we are in an eigenstate of Q time evolution by Schrödinger will not change that → symmetry = cons. law

$U = e^{-iaQ}$, Q is the generator of the sym

e.g. free particle on a line: $H = \frac{p^2}{2m}$, $[H, p] = 0$

momentum is conserved → translation sym

$U = e^{-iap/\hbar}$ p generates translations → $U(a)|x\rangle = |x+a\rangle$

~~for an eigenstate~~. acts on w.f. as $U(a)\psi(x) = \psi(x-a)$

H is gen of time-translations → energy conserved

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so if we ~~know~~ want to study a rot-inv. system in 3d → ang. mom. conserved

rotations: $U(\theta) = e^{-i\vec{\theta}\cdot\vec{J}/\hbar}$ → ~~the~~ generator are ang. mom. ops J_i

~~we want to st~~

What is all this ang. mom stuff?

We want to understand how the algebra of J's is realized on $\mathcal{H}$

and what this means for the structure of $\mathcal{H}$

{ this is the same thing as saying we are studying the rep. theory of a Lie algebra $SU(2)$ }

for a particle in 3d this ang. mom. algebra is realized on the $\mathcal{H}$ (where we have $[x_i, p_j] = i\hbar\delta_{ij}$)

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & -\theta \\ \theta & 1 \end{pmatrix}$$

infinitesimal rotations are generated by $\vec{L} = \vec{r} \times \vec{p}$

$L_z = x p_y - y p_x$

these satisfy the algebra $[L_i, L_j] = i\hbar\epsilon_{ijk}L_k$

further $L^2 = \sum L_i^2$ is the Casimir $[L^2, L_i] = 0$ $\Rightarrow$ statement that L_i's don't commute? ang. mom. vectors do not exist

so a possible CSCO might be L^2, L_z i.e. label states by eigenvalues $|l, m\rangle$

define raising/lower ops $L_\pm = L_x \pm i L_y$ where $L_\pm^\dagger = L_\mp$

↳ because $[L_z, L_\pm] = \pm\hbar L_\pm$ →

$$L_\pm |lm\rangle \Rightarrow L_z L_\pm |lm\rangle = (\pm\hbar L_\pm + L_\pm L_z)|lm\rangle = L_\pm(\hbar(m\pm 1)|lm\rangle)$$

$$\text{i.e. } L_z(L_\pm|lm\rangle) = \hbar(m\pm 1)(L_\pm|lm\rangle)$$

eigenstate w $m \pm 1$

$$\rightarrow L_\pm|lm\rangle = *|l\, m\pm 1\rangle$$

(just like $[N, a^\dagger] = a^\dagger$)

Algebra of ang. mom. ops

$$[L_i, L_j] = i\hbar\epsilon_{ijk}L_k \quad \text{and} \quad [L^2, L_i] = 0$$

$$L^2|lm\rangle = \hbar^2 l(l+1)|lm\rangle$$

$$L_z|lm\rangle = \hbar m|lm\rangle$$

$m = -l \ldots l$

$L_\pm = L_x \pm i L_y$ w/ $[L_\pm, L^2] = 0$, $[L_+, L_-] = 2\hbar L_z$

$$L^2 = L_+L_- + L_z^2 - \hbar L_z$$

$$L_\pm|lm\rangle = \hbar\sqrt{l(l+1) - m(m\pm 1)}\,|l\, m\pm 1\rangle$$

How does this all relate to w.f. / Schrödinger's eq?

Recall in 3d $\quad H = -\frac{\hbar^2}{2m}\nabla^2 + V(r)$

the Laplacian is $\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\left(\frac{\partial^2}{\partial\phi^2}\right)$

we want to find solutions to SE of the form: $\psi(r,\theta,\phi) = R(r)\,Y(\theta,\phi)$

$$HRY = ERY$$

$$\frac{-\hbar^2}{2m}\left(\frac{Y}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) + \frac{R}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{R}{r^2\sin^2\theta}\left(\frac{\partial^2 Y}{\partial\phi^2}\right)\right) + VRY = ERY$$

$$\underbrace{\left(\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) - \frac{2mr^2}{\hbar^2}V(r) + \frac{2mr^2}{\hbar^2}E\right)}_{\text{Radial eq.} \;=\; \text{const}} + \underbrace{\frac{1}{Y}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2}\right)}_{\text{angular e.q.} \;=\; -\text{const}} = 0$$

denote const $= \ell(\ell+1)$

angular eq. $\quad \frac{1}{Y}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2}\right) = -\ell(\ell+1)$

solutions to this eq. are Spherical harmonics $\quad Y_{\ell m}(\theta,\phi)$

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in braket language we are looking for sols $\quad |xyz\rangle = |r\rangle \otimes |\theta\phi\rangle$

Writing L^2 in spherical coords $\quad$ i.e. $L_z = -i\hbar(x\partial_y - y\partial_x) = -i\partial_\phi$

$$L^2 = -\hbar^2\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial}{\partial\phi^2}\right)$$

hence angular eq. is $\quad L^2 Y_{\ell m}(\theta,\phi) = \hbar^2\ell(\ell+1)\,Y_{\ell m}(\theta,\phi)$

the spherical harmonics are the angular momentum eigenstates

$$Y_{\ell m}(\theta,\phi) \equiv \langle\theta,\phi|\ell m\rangle \quad \rightarrow \quad L^2|\ell m\rangle = \hbar^2\ell(\ell+1)\,|\ell m\rangle$$

The radial eq.

$$\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) - \frac{2mr^2}{\hbar^2}(V(r) - E)R = l(l+1)R$$

is a little trickier, depending on the $V(r)$

let's try ∞-spherical well $V(r) = \begin{cases} 0 & r<a \\ \infty & r>a \end{cases}$

inside

$$\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) + \frac{2mr^2}{\hbar^2} E R = l(l+1)R$$

what about $l=0$ solution: $\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) = -\frac{2mE}{\hbar^2} R$ (*)

Consider change of variables: $u(r) \equiv r R(r)$

$R = \frac{u}{r}$, $\frac{dR}{dr} = \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2}$,

so $\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) = \frac{d}{dr}\left(r\frac{du}{dr} - u\right) = r\frac{d^2u}{dr^2}$

$\Rightarrow$ (*) $\boxed{\frac{d^2u}{dr^2} = -k^2 u}$ $k = \frac{\sqrt{2mE}}{\hbar}$

$$R(r) = \frac{u(r)}{r} = \frac{A\sin(kr) + B\cos(kr)}{r}$$

finite as $r \to 0$? $B=0$, we also want $u(a)=0$

$\sin ka = 0 \to \underline{ka = n\pi}$

$k = \frac{\sqrt{2mE}}{\hbar} \to E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}$ are allowed energies (just as in 1d)

~~Example 1~~ ∞-well (from $\int d^3r\, |\psi|^2$
$\hookrightarrow dr\, d\theta\, d\phi\, r^2 \sin\theta$

normalize: $\int_0^\infty dr\, r^2 R^2 = 1$

$\int_0^a dr\, A^2 \sin^2 kr = \frac{a}{2} - \frac{\sin(2ak)}{4k} \to 0$ ~~$\to 2A^2 \sqrt{\frac{2}{a}}$~~ $A = \sqrt{\frac{2}{a}}$

hence $\psi_{n00} = \frac{1}{\sqrt{2\pi a}} \frac{\sin(n\pi r/a)}{r}$ as $Y_{00} = \frac{1}{\sqrt{4\pi}}$

Hydrogen like potential: Consider 3d particle in $V \sim -\frac{1}{r}$

Consider the following state in superposition of energy eigenstates $\psi_{n\ell m}$, $E_n = \frac{E_1}{n^2}$

~~$|\psi\rangle = \frac{1}{6}(4|1$~~ $\psi = \frac{1}{6}\left(4\psi_{100} + 3\psi_{211} - \psi_{210} + \sqrt{10}\,\psi_{21-1}\right)$

i) what is $\langle E\rangle$ $\to |\psi\rangle = \frac{1}{6}\left(4|100\rangle + 3|211\rangle - |210\rangle + \sqrt{10}\,|21{-1}\rangle\right)$

$$\langle E\rangle = \left(\frac{4}{6}\right)^2 E_1 + \left(\frac{3}{6}\right)^2 E_2 + \left(-\frac{1}{6}\right)^2 E_2 + \left(\frac{\sqrt{10}}{6}\right)^2 E_2$$

$$= \frac{16}{36} E_1 + \frac{9+1+10}{36}\frac{1}{4} E_1 = \frac{21}{36} E_1$$

ii) $\langle L^2\rangle$? $L^2 |n\ell m\rangle = \hbar^2 \ell(\ell+1)|n\ell m\rangle$

$$\langle L^2\rangle = \hbar^2\left(\frac{1}{36}\left(4^2\cdot 0 + 3^2\cdot 2 + 2 + 10\cdot 2\right)\right) = \hbar^2 \frac{10}{9}$$

iii) $\langle L_z\rangle = ?$ $L_z|n\ell m\rangle = \hbar m|n\ell m\rangle$

$$\langle L_z\rangle = \frac{\hbar}{36}\left(9 + 10\cdot(-1)\right) = -\frac{\hbar}{36}$$

Consider the electron in the g.s. of hydrogen (1,0,0) state

$$\Psi(\vec{r},t) = \psi_{100}(\vec{r})\, e^{-iE_1 t/\hbar}$$

i) is it in motion?

$$\langle x \rangle = \int d^3r\, \Psi^* x \Psi$$

x does not vary w/ time so it's best to say the e^- is not in motion

→ stationary state

Density operators

Recall that a projector onto a state $|\psi\rangle$ is $\mathbb{P}_\psi = |\psi\rangle\langle\psi|$

if the system is in state $|\psi\rangle$ we can compute expectation vals of any op in $\mathcal{H}$

as $\langle \mathcal{O}\rangle = \langle\psi|\mathcal{O}|\psi\rangle = \mathrm{tr}(\mathcal{O}\,\mathbb{P}_\psi)$

what if we use this to define expectation values

$\mathrm{tr}(\mathcal{O}\rho) = \langle\mathcal{O}\rangle$ where ρ the density matrix is a more general state than $|\psi\rangle$

For a <u>pure state</u>: $\rho_\psi = \mathbb{P}_\psi = |\psi\rangle\langle\psi|$

but we can imagine states like $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$ <u>mixed states</u>

like an ensemble of states — p_i: prob of being in $|\psi_i\rangle$

but more general than a superposition

also note: no phases

<u>Properties</u>:
1. unit trace: $\mathrm{tr}\,\rho = 1$ (i.e. probs add to one)
2. Hermitian: $\rho^\dagger = \rho$ (meaning the eigenvalues are probabilities)
3. ρ is positive all eigenvalues are real and <u>non negative</u> (probs)

$$\rho = \sum_a p_a |\psi_a\rangle\langle\psi_a| \quad \text{w/ } 0 \le p_a \le 1 \text{ and } \sum_a p_a = 1$$

QM vs. CM uncertainty:

$$\rho_M = p_1 |\psi_1\rangle\langle\psi_1| + p_2 |\psi_2\rangle\langle\psi_2|$$

vs. $|\phi\rangle = a|\psi_1\rangle + b|\psi_2\rangle \rightarrow \rho_P = |a|^2 |\psi_1\rangle\langle\psi_1| + |b|^2 |\psi_2\rangle\langle\psi_2| + ab^* |\psi_1\rangle\langle\psi_2| + \text{h.c.}$

~~Consider:~~

Example: $|+\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle)$ is an eigenstate of σ_x so while it's σ_z value is uncertain, the σ_x value is

the mixed state where $|\uparrow\rangle$ and $|\downarrow\rangle$ occur w/ equal prob.

$$\rho_{mixed} = \tfrac{1}{2}|\uparrow\rangle\langle\uparrow| + \tfrac{1}{2}|\downarrow\rangle\langle\downarrow| \qquad (\text{cannot be written as } |\psi\rangle\langle\psi|)$$

$$= \tfrac{1}{2}\mathbb{1} \qquad \to \text{ensemble where } |\uparrow\rangle \text{ and } |\downarrow\rangle \text{ occur w/ equal prob.}$$

very different from pure state $|+\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle)$ i.e. superposition of $\uparrow$ $\downarrow$

$$\rho_{pure} = |+\rangle\langle+| = \tfrac{1}{2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} \qquad \text{obviously different}$$

but consider the measurement of σ^x → the projection onto $|+\rangle$

i.e. prob we get $\uparrow$ if we measure σ^x

$$\mathbb{P}_+ = |+\rangle\langle+|$$

$$\langle \mathbb{P}_+ \rangle_{\rho_{pure}} = \text{tr}\, \rho_{pure}\, \mathbb{P}_+ = \langle+|+\rangle\langle+|+\rangle = 1$$

obvs, we are certain we get $\uparrow$ if we measure σ^x in state $|+\rangle$

$$\langle \mathbb{P}_+ \rangle_{\rho_{mixed}} = \text{tr}\, \rho_{mixed}\, \mathbb{P}_+ = \frac{1}{2} \qquad \text{we get a random result}$$

i.e. in this mixed state measure on any x, y, z → random

Purity Testing

Purity consider the pure state: $|\psi\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + e^{i\alpha}|1\rangle\right)$

and mixed state: $\rho_\beta = \frac{1}{2}\left(|0\rangle\langle 0| + |1\rangle\langle 1|\right)$

i) write density matrix of $|\psi\rangle$

$$\rho_\psi = |\psi\rangle\langle\psi| = \frac{1}{2}\left(|0\rangle\langle 0| + |1\rangle\langle 1| + e^{-i\alpha}|0\rangle\langle 1| + e^{i\alpha}|1\rangle\langle 0|\right) = \frac{1}{2}\begin{pmatrix} 1 & e^{-i\alpha} \\ e^{i\alpha} & 1 \end{pmatrix}$$

ii) compute $\langle \sigma_z \rangle$

$$\langle \sigma_z \rangle_\psi = \langle\psi|\sigma_z|\psi\rangle = \frac{1}{2}\left(\langle 0|\sigma_z|0\rangle + \langle 1|\sigma_z|1\rangle\right) = \frac{1}{2} - \frac{1}{2} = 0$$

$$\langle \sigma_z \rangle_{\rho_\beta} = \mathrm{Tr}\, \rho_\beta \sigma_z = \mathrm{tr}\begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{in } z\text{-basis}$$

$$= 0$$

iii) Define purity of a state $\mathrm{Tr}(\rho^2)$

If ρ is pure, $\mathrm{Tr}(\rho^2)$? $\rightarrow \rho^2 = \rho \quad \mathrm{Tr}(\rho) = 1$

let's check: $\mathrm{tr}\, \rho_\psi^2 = \mathrm{tr}\, \rho_\psi = 1$

$\mathrm{tr}\, \rho_\beta^2 = \frac{1}{4}\mathrm{tr}\, \mathbb{1} = \frac{1}{2}$ ← maximally mixed

iv) compute $\langle \sigma_x \rangle = \mathrm{tr}\, \rho\, \sigma_x$ → compute in $|\pm\rangle$ basis

$$\langle \sigma_x \rangle_\psi = \langle +|\rho_\psi \sigma_x|+\rangle + \langle -|\rho_\psi \sigma_x|-\rangle = \langle +|\rho_\psi|+\rangle - \langle -|\rho_\psi|-\rangle$$

$$= \frac{1}{2}\mathrm{tr}\begin{pmatrix} 1 & e^{-i\alpha} \\ e^{i\alpha} & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{1}{2}\mathrm{tr}\begin{pmatrix} e^{-i\alpha} & 1 \\ 1 & e^{i\alpha} \end{pmatrix} = \frac{1}{2}\left(e^{i\alpha} + e^{-i\alpha}\right) = \cos\alpha$$

$$\langle \sigma_x \rangle_{\rho_\beta} = \frac{1}{2}\mathrm{tr}(\mathbb{1}\,\sigma_x) = 0$$