A reversible framework for entanglement and other resource theories

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What this talk is about:

1. An *analogy* of entanglement theory and the second law of thermodynamics

2. Using the analogy to solve *open problems* in entanglement theory

3. Extending the analogy to *almost every* resource theory

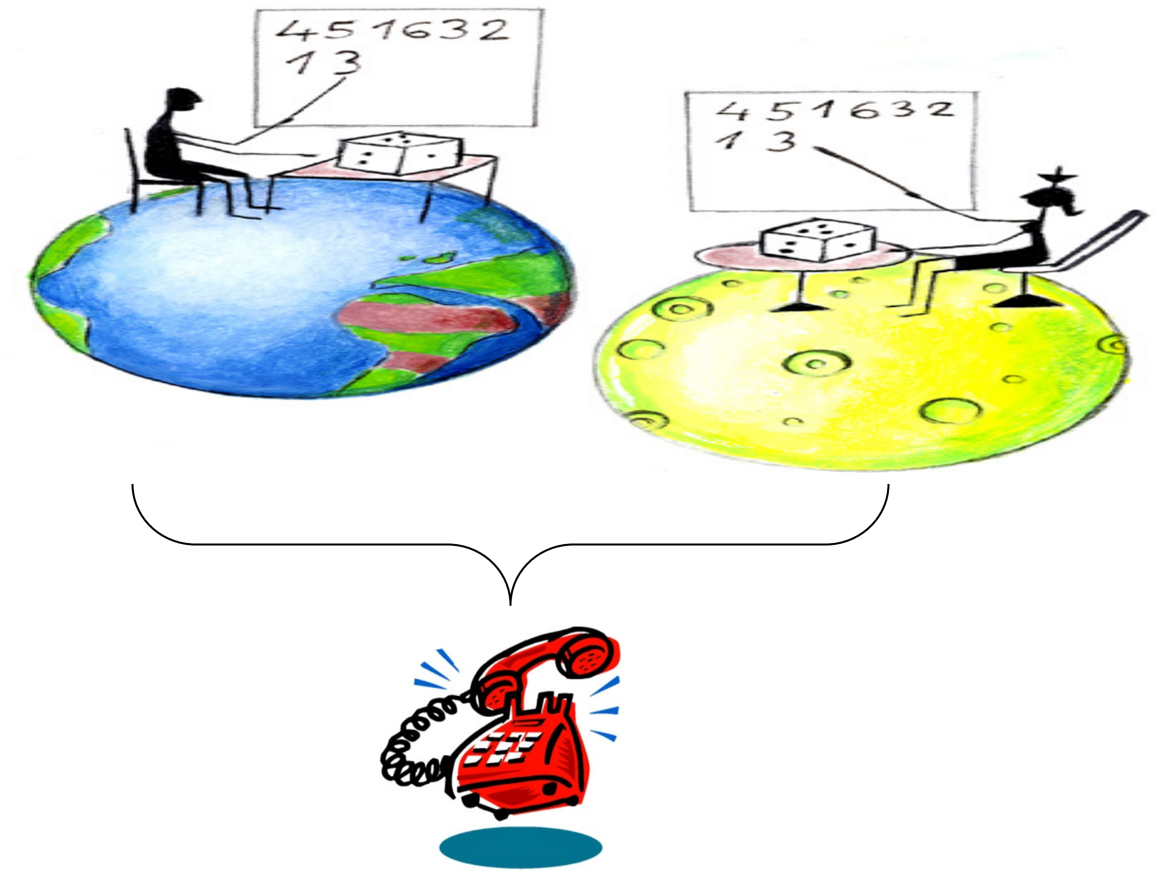
4. Implications to *Bell inequalities* and its statistical significance

What this talk is not about:

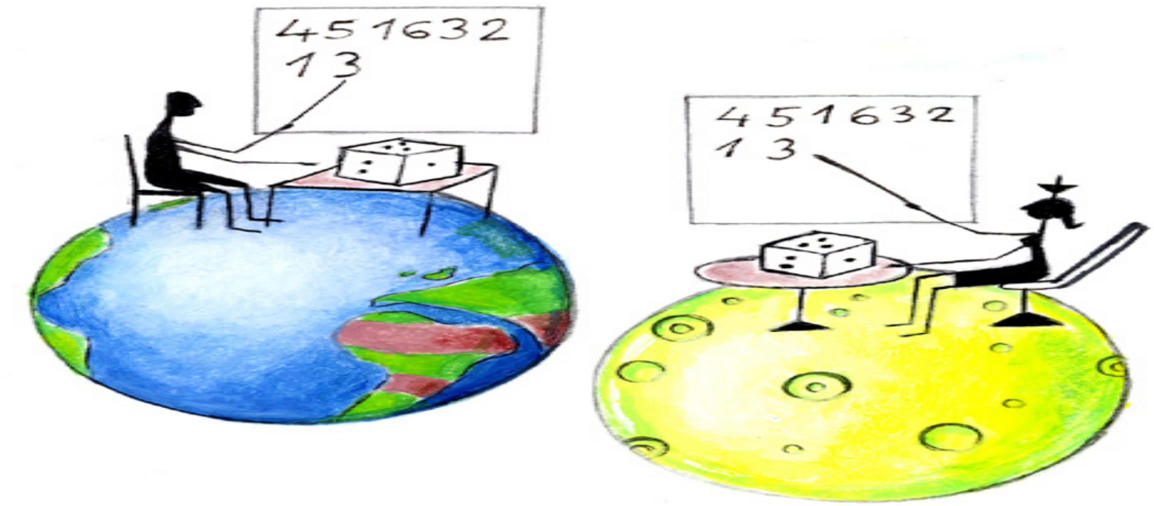
1. ~~Drawing work from entangled states~~
2. ~~Thermal entanglement in equilibrium systems~~
3. ~~ETC….~~

In particular, no knowledge of *real* thermodynamics is assumed or needed…

Quantum Entanglement

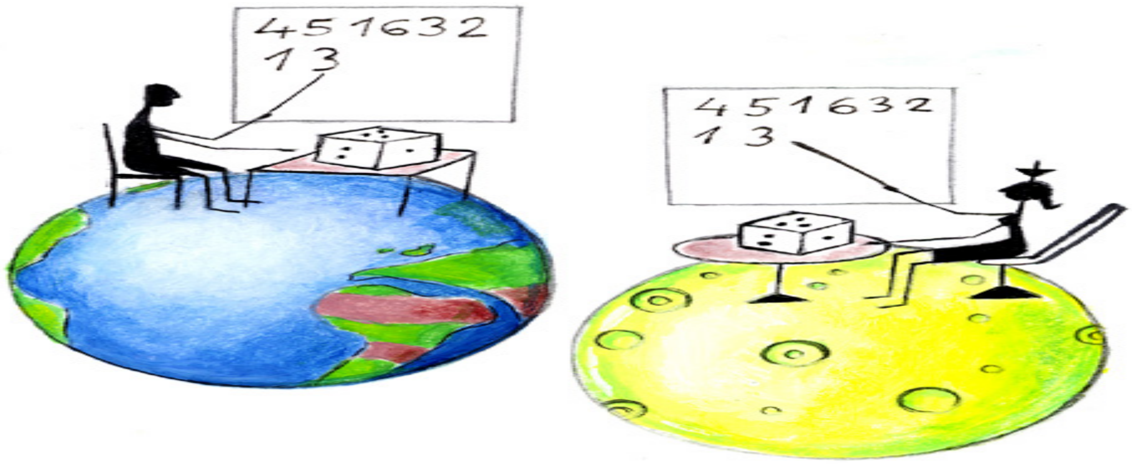


Quantum Entanglement



$$\underbrace{\hphantom{xxxxxxxxxxxxxxxx}}_{\displaystyle \sum_i p_i \rho_A^{\,i} \otimes \rho_B^i}$$

Quantum Entanglement



$$\rho \neq \sum_i p_i \rho_1^i \otimes \ldots \otimes \rho_k^i$$

Entanglement is

What cannot be created by local operations and classical communication (LOCC)

Restricted Operations and Resources

- In Physics we commonly deal with restrictions on the physical processes/operations available. Entanglement theory is an instance of such paradigm.

Restricted set of operations	Free states	Resource
Local operations and classical communication (LOCC)	Separable states	Entanglement

The Holy Grail of Resource Theories

- Given some set of allowed operations on a physical system, which transformations from one state of the system into another can be realized?

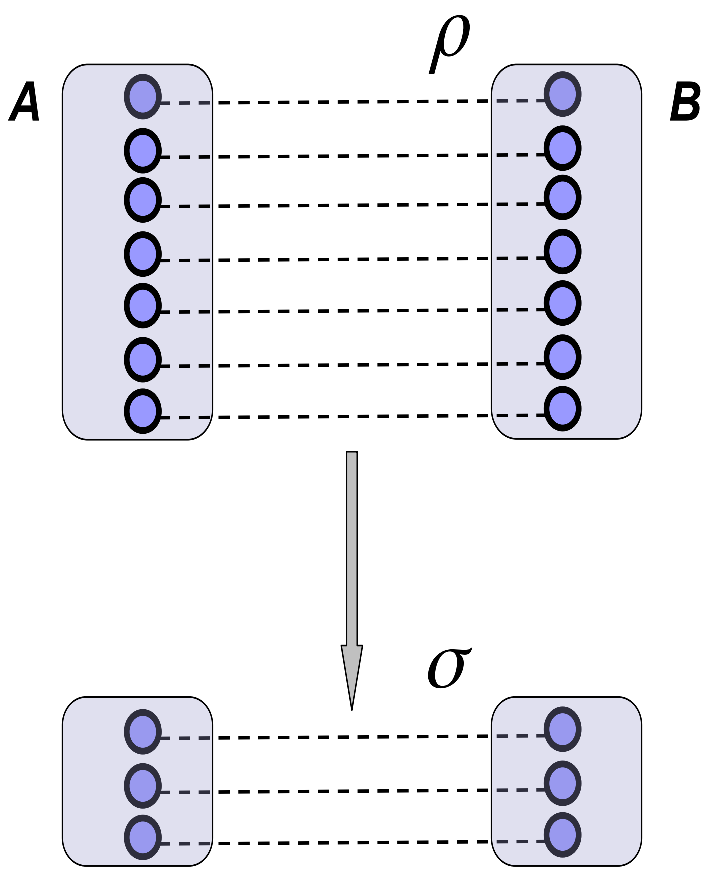
When and how can we transform a resource from one form into another?

LOCC transformations

- Given two quantum states, can we transform one into the other by LOCC?

- At what *rate* can we make the transformation?

- We will be interested in this question in the limit of a large number of copies of the two states

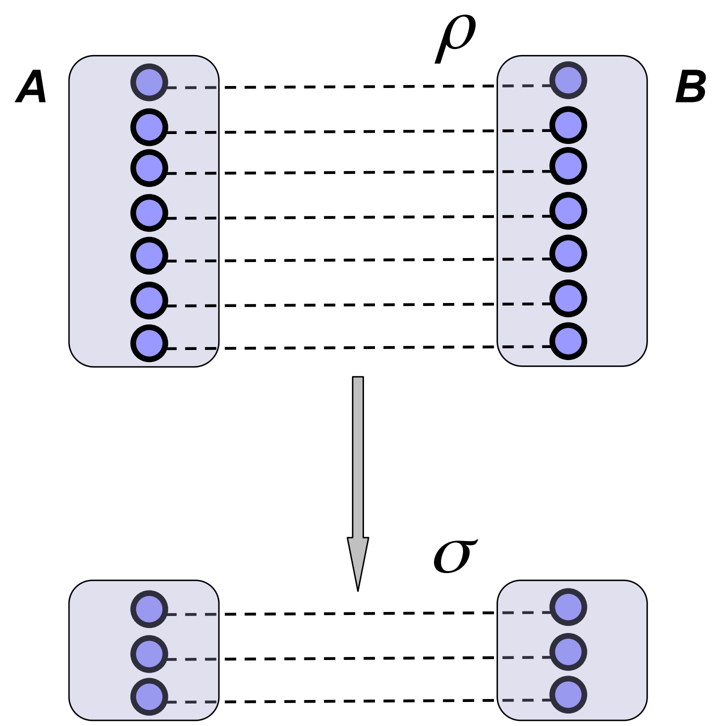


$$\rho^{\otimes n} \rightarrow_{LOCC} \sigma_n \approx \sigma^{\otimes k_n}$$

LOCC asymptotic entanglement transformations

Optimal rate of conversion:

How many copies of rho we have to invest per copy of sigma?



Mathematically:

$$R(\rho \to \sigma) = \inf\left\{\frac{n}{m} : \lim_{n\to\infty}\left(\min_{\Lambda\in LOCC} \| \Lambda(\rho^{\otimes n}) - \sigma^{\otimes m} \|_1\right) = 0\right\}$$

Bipartite pure state entanglement transformations

- The simplest form of entanglement: $|\psi\rangle_{AB} = \sum_k \lambda_k |k\rangle_A \otimes |k\rangle_B$

- (Bennett et al 96) Transformations are *reversible*

$$|\psi\rangle^{\otimes n} \rightarrow_{LOCC} |\varphi\rangle^{\otimes nE(\psi)/E(\varphi)} \rightarrow_{LOCC} |\psi\rangle^{\otimes n}$$

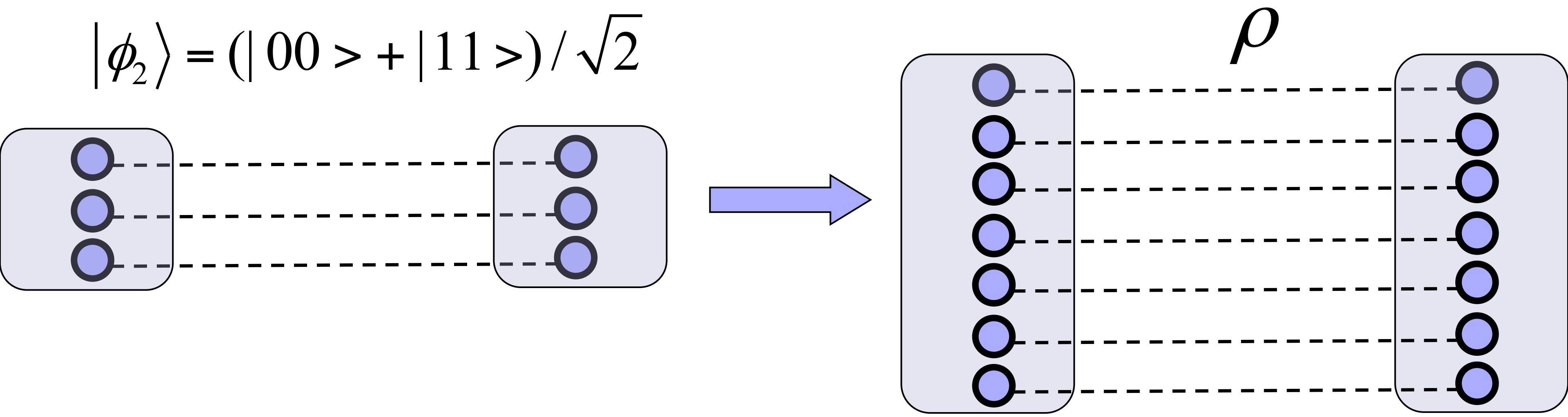
with $E(\rho) = S(\rho_A)$ the unique measure of entanglement

- Equivalent to the existence of a *total order*

$$|\psi\rangle^{\otimes n} \rightarrow_{LOCC} |\varphi\rangle^{\otimes n}$$ if, and only if, $$E(\psi) \geq E(\varphi)$$

Mixed state entanglement

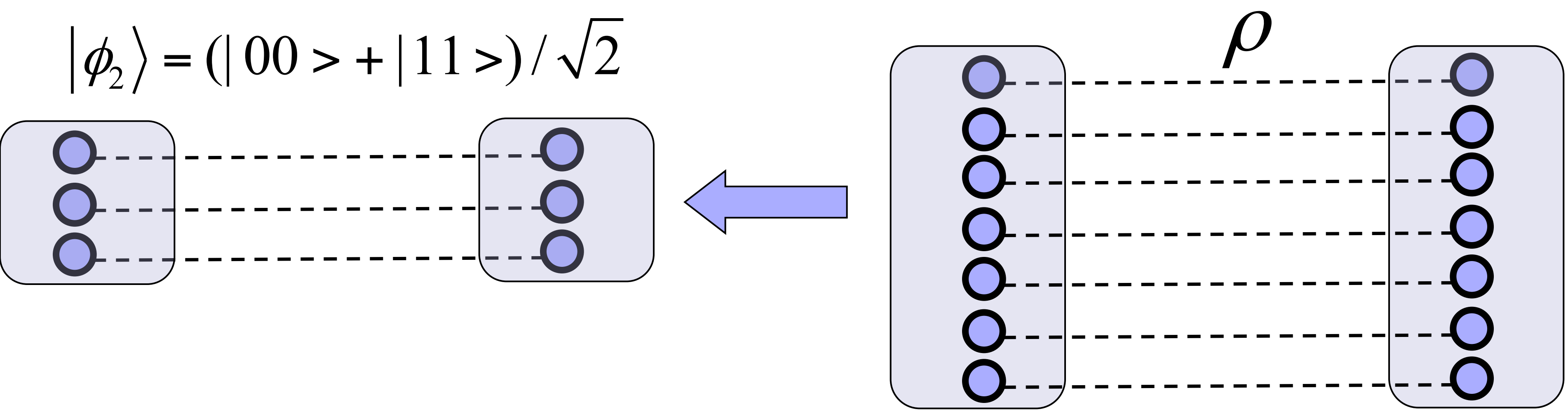
- Entanglement cost: $E_C(\rho) = R(\phi_2 \rightarrow \rho)$



Mixed state entanglement

- Distillable entanglement: $$E_D(\rho) = R(\rho \rightarrow \phi_2)^{-1}$$

$$|\phi_2\rangle = (|00> + |11>)/\sqrt{2}$$



Mixed state entanglement

Manipulation of mixed state entanglement under LOCC is *irreversible*

- (3 x Horodecki 98) In general:

$$E_C(\rho) > E_D(\rho)$$

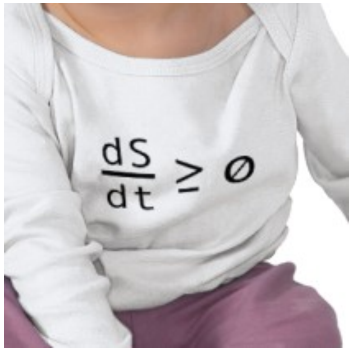
- Extreme case, *bound* entanglement:

$$E_C(\rho) > 0, E_D(\rho) = 0$$

- No unique measure for entanglement manipulation under LOCC. In fact there is a whole zoo of them...

Thermodynamics and its second law

• (Clausius, Kelvin, Planck, Caratheodory) **The second law of thermodynamics:** The entropy of a system in thermal equilibrium can never decrease during an adiabatic process (a transformation which doesn't involve exchange of heat with environment)



• In 1999 Lieb and Yngvason proposed an axiomatic approach to the second law, which try to improve on the non-rigorous arguments of the previous formulations.

•It's in the *mathematical structure* they identified which we are most interested!

Lieb and Yngvason Approach to 2nd Law

- X, Y, Z are thermodynamical states
- X $\Rightarrow$ Y : there is an adiabatic transformation from X to Y
- (X, Y): composition of the two systems X and Y
- *t* X: system composed of *t* times X.

- ## Axioms:

1. X $\Rightarrow$ X
2. If X $\Rightarrow$ Y and Y $\Rightarrow$ Z then X $\Rightarrow$ Z
3. X $\Rightarrow$ Y and X' $\Rightarrow$ Y' then (X, X') $\Rightarrow$ (Y, Y')
4. X $\Rightarrow$ Y then *t*X $\Rightarrow$ *t*Y for all *t*
5. X $\Rightarrow$ (*t* X, *(1-t)*X) for all 0 < *t* < 1
6. If (X, *s Z*) $\Rightarrow$ (Y, *s* Z) for s $\to$ 0 then X $\Rightarrow$ Y

- ## Comparison Hypothesis:

For all X, Y either X $\Rightarrow$ Y or Y $\Rightarrow$ X

Lieb and Yngvason Formulation of the 2nd Law

(Lieb and Yngavason 99) Axioms 1-6 and the Comparison Hypothesis imply that there is a function *S* on the state space such that

X $\Rightarrow$ Y if, and only if, S(X) <= S(Y)

The function *S* can be taken to be *additive, i.e.*

$$S(\,(t\,X,\; s\,Y)\,) = t\,S(X) + s\,S(Y)$$

and then it is *unique* up to affine transformations

Entanglement Theory and the Second Law

- The LY formulation of the 2^{nd} law is formally equivalent to the law of transformations of bipartite pure states by LOCC

- All LY axioms are true for bipartite pure states

- But for general (mixed) states, the analogy breaks down: there is bound entanglement...

Entanglement Theory and the Second Law

- Work by Horodecki&Oppenheim, Popescu&Rorhlich, Plenio&Vedral, Vedral&Kashefi explored this connection through several angles and asked whether one could somehow find a thermodynamical formulation for general states

- BTW: Thinking about this had already led to technical progress in entanglement theory: The Horodeckis discovered *entanglement activation* inspired by the thermodynamical analogy...

Entanglement in Fantasy Land

• We will look at a thermodynamical formulation for entanglement theory by extending the class of operations allowed *beyond* LOCC

• Extending the class of operations to get a more tractable theory is not a new idea (Rains, Eggeling et al, Audenaert et al, Horodecki et al):

1. Separable operations
2. PPT operations
3. LOCC + bound entanglement

Non-entangling maps

• We will consider the *extreme* situation, and consider the manipulation of entanglement by *all transformations which don't generate entanglement*

• The class of non-entangling maps consists of all quantum operations which maps separable states to separable states

Reversibility under Non-entangling Maps

- (B, Plenio 08: Informal) Under the class of (asymptotic) non-entangling operations, entanglement theory is reversible. There is an entanglement measure *E* such that for every all states rho, sigma,

$$\rho^{\otimes n} \to \sigma^{\otimes n E(\rho)/E(\sigma)} \to \rho^{\otimes n}$$

and, equivalently,

$$\rho^{\otimes n} \to \sigma^{\otimes n} \text{ if, and only if, } E(\rho) \geq E(\sigma)$$

Two measures of entanglement

- Relative entropy of entanglement:

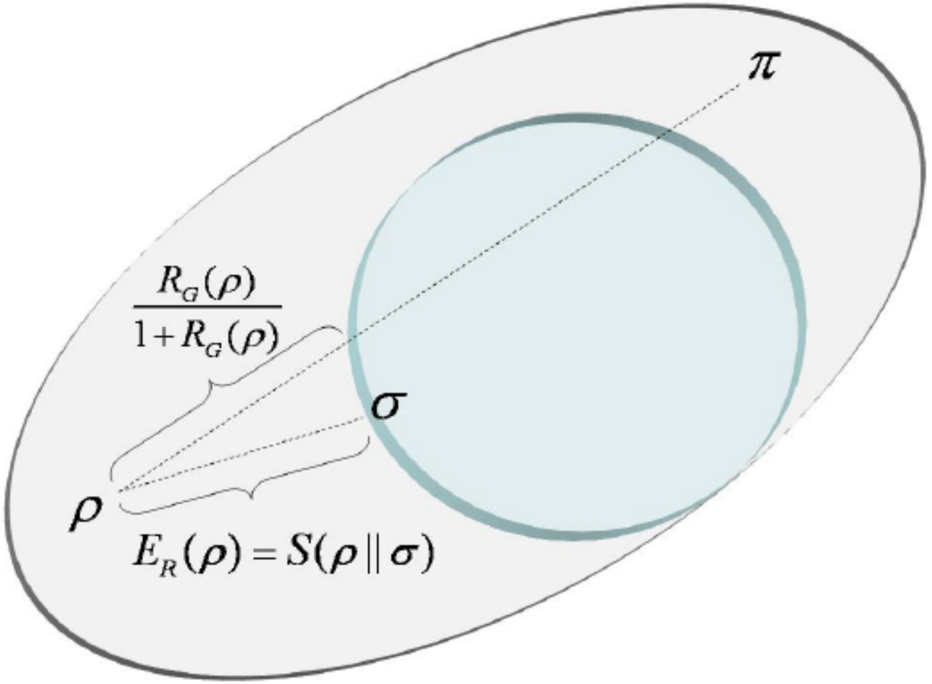
$$E_R(\rho) = \min_{\sigma \in S} S(\rho \| \sigma) \qquad S(\rho \| \sigma) = tr(\rho(\log \rho - \log \sigma))$$

Vedral&Plenio 97

- Robustness of Entanglement:

$$r_G(\rho) = \min \ s : \frac{\rho + s\pi}{1+s} \in S$$

Vidal&Tarrach 99 and Harrow&Nielsen 03



Asymptotically non-entangling operations

- Definition: A quantum operation Λ is ε -non-entangling if

$$r_G(\Lambda(\sigma)) \leq \varepsilon$$

For every separable state σ

- We say that a sequence of maps $\{\Lambda_n\}$ is asymptotically non-entangling if each Λ_n is ε_n-non-entangling and

$$\lim_{n\to\infty} \varepsilon_n = 0$$

Optimal rate of conversion

- Optimal rate of conversion under asymptotically non-entangling operations:

$$R(\rho \to \sigma) = \inf\left\{\frac{n}{m} : \lim_{n\to\infty}\left(\min_{\Lambda\in NE(\varepsilon_n)} \| \Lambda(\rho^{\otimes n}) - \sigma^{\otimes m} \|_1\right) = 0,\ \lim_{n\to\infty}\varepsilon_n = 0\right\}$$

- $NE(\varepsilon)$ denotes the class of ε - resource non-generating operations

Reversibility Again

For every quantum states ρ, σ

$$R(\rho \to \sigma) = E_R^{\infty}(\sigma) / E_R^{\infty}(\rho)$$

$$E_R^{\infty}(\rho) = \lim_{n \to \infty} \frac{E_R(\rho^{\otimes n})}{n}$$

Implies: $\rho^{\otimes n} \to \sigma^{\otimes n}$ *iff* $E_R^{\infty}(\rho) \geq E_R^{\infty}(\sigma)$

Applications

- Our result has many applications to the *LOCC* manipulation of entanglement:

1. (B., Plenio 09) Proof that $R_{LOCC}(\rho \to \sigma)$ is positive for all entangled sigma

2. (Beigi, Shor 09) Proof that the set of PPT states give no useful approximation to the set of separable states

3. (B. 10) Proof that no finite set of three qubit states forms a Minimum-Reversible-Entanglement-Generating-Set

4. (B., Horodecki 10) Progress characterizing all mixed states which are reversible: they all consist of ensembles of pure states perfectly distinguishable by separable measurements.

Connection to LY approach

- The structure form of our theorem is the same as LY formulation to the second Law. However, are all of his axioms true?

- Axioms:
1. $X \Rightarrow X$ ✓
2. If $X \Rightarrow Y$ and $Y \Rightarrow Z$ then $X \Rightarrow Z$ ✓
3. $X \Rightarrow Y$ and $X' \Rightarrow Y'$ then $(X, X') \Rightarrow (Y, Y')$?
4. $X \Rightarrow Y$ then $tX \Rightarrow tY$ for all t ✓
5. $X \Rightarrow (t\,X, (1\text{-}t)X)$ for all $0 < t < 1$ ✓
6. If $(X, s\,Z) \Rightarrow (Y, s\,Z)$ for $s \to 0$ then $X \Rightarrow Y$ ✓

- Comparison Hypothesis:

For all X, Y either $X \Rightarrow Y$ or $Y \Rightarrow X$ ✓

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  - Comparison Hypothesis:

  For all X, Y either $X \Rightarrow Y$ or $Y \Rightarrow X$ ✓

- The swap and the identity are non-entanling, but their tensor product is not! Not directly a counterexample, but...

Connection to LY approach

- The potential failure of axiom 3 is equivalent to the non-additivity of the unique entanglement measure in the theory

- Open question: Is $E_R^{\infty}(\rho)$ additive?

- Interesting from the viewpoint of understanding the necessity of LY axioms

- Would also have consequences to quantum complexity theory: (B., Horodecki 10) Additivity implies that QMA(k) = QMA(2) for all *k*>2

Is entanglement special?

• The result can actually be extended far more broadly than to entanglement. Consider a general resource theory:

Restricted set of operations	Free states	Resource

• Let M be the set of free states in the theory of M-resources

• Assume M is closed and convex (the theory allows for mixing)

Two measures of resource

- The relative entropy of *M*-resource is given by

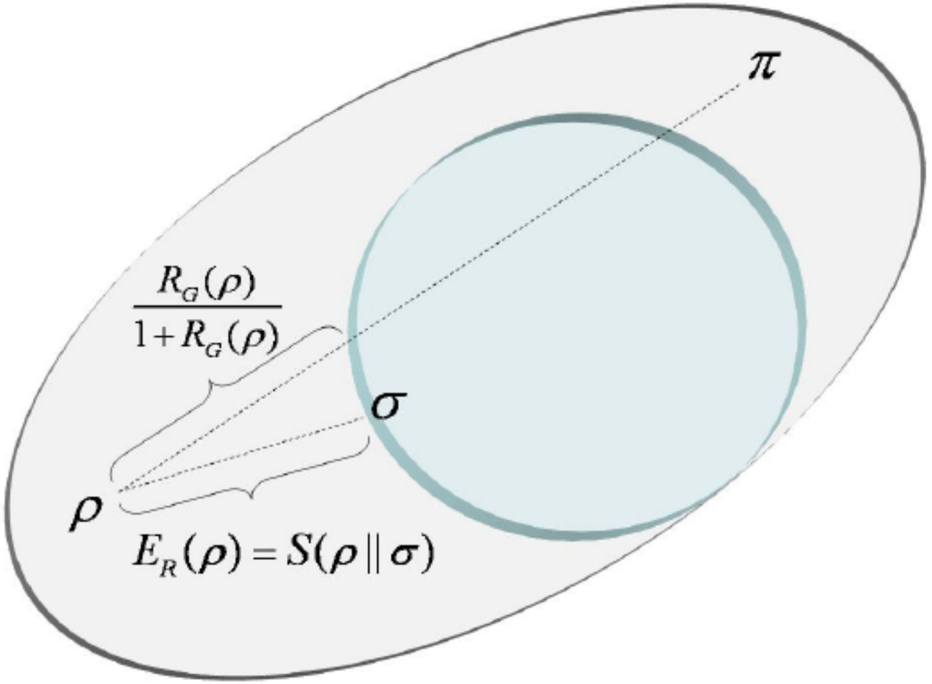
$$E_M(\rho) = \min_{\sigma\in M} S(\rho\|\sigma) \qquad S(\rho\|\sigma) = tr(\rho(\log\rho - \log\sigma))$$

Vedral&Plenio 97

- The robustness of *M*-resource is given by

$$r_M(\rho) = \min\ s : \frac{\rho + s\pi}{1+s} \in M$$

Vidal&Tarrach 99 and Harrow&Nielsen 03



Asymptotically resource non-generating operations

- Definition: A quantum operation Λ is ε -resource non-generating if

  $r_M(\Lambda(\sigma)) \leq \varepsilon$ for every non-resource state σ

- We say that a sequence of maps $\{\Lambda_n\}$ is asymptotically resource non-generating if each Λ_n is ε_n-resource non-generating and

$$\lim_{n\to\infty} \varepsilon_n = 0$$

Reversibility for Resource Theories

Under a few assumptions on *M*, for every quantum states ρ, σ

$$\boxed{R(\rho \to \sigma) = E_M^{\infty}(\sigma) / E_M^{\infty}(\rho)}$$

$$E_M^{\infty}(\rho) = \lim_{n \to \infty} \frac{E_M(\rho^{\otimes n})}{n}$$

Implies: $\rho^{\otimes n} \to \sigma^{\otimes n}$ *iff* $E_M^{\infty}(\rho) \geq E_M^{\infty}(\sigma)$

The main idea

- The main idea is to connect the *convertibility* of resource states to the *distinguishability* of resource states from non-resource ones

- Basically, if a resource theory is such that we can distinguish, by measurements, many copies of a resource state from non-resource states *pretty well,* then the theory is reversible under the class of resource non-generating operations!

The main idea

- **Quantum Hypothesis Testing**: given several copies of a quantum state and the promise that you are given either ρ (null hypothesis) or σ (alternative hypothesis), decide which you have

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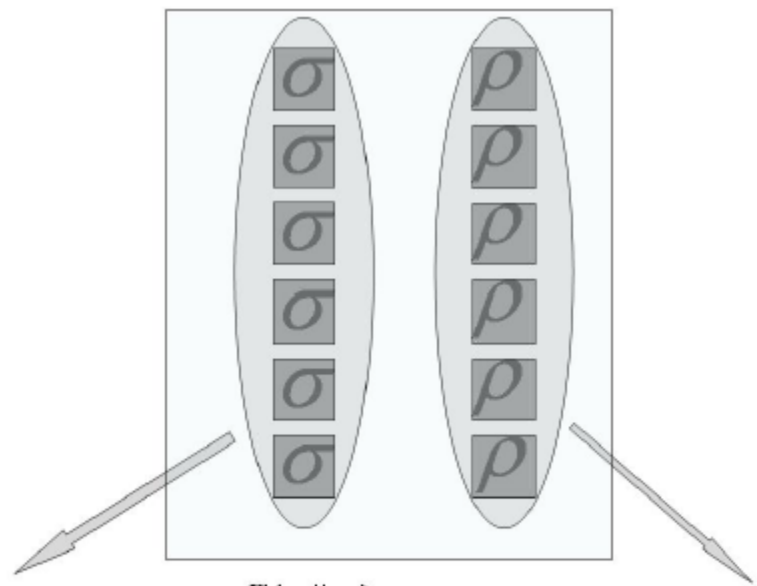
- **Quantum Hypothesis Testing**: given several copies of a quantum state and the promise that you are given either ρ (null hypothesis) or σ (alternative hypothesis), decide which you have

- **Quantum Stein's Lemma**:

$$\beta_1(A_n) := tr(\rho^{\otimes n}(I - A_n))$$

$$\beta_2(A_n) := tr(\sigma^{\otimes n} A_n)$$



$$\beta_2(n) \approx 2^{-nS(\rho\|\sigma)} \qquad \beta_1(n) \to 0$$

Hiai&Petz 91 and Ogawa&Nagaoka 99

The main idea

- ***Resource* Hypothesis Testing**: given a sequence of quantum states ω_n acting on $H^{\otimes n}$, with the promise that either $\{\omega_n\}_{n\in\mathbb{N}}$ is a sequence of unknown non-resource states or $\omega_n = \rho^{\otimes n}$, for some resource state ρ, decide which is the case.

Probabilities of Error:

$$\beta_1(A_n) := tr(\rho^{\otimes n}(I - A_n))$$

$$\beta_2(A_n) := \max_{\omega_n \in M} tr(\omega_n A_n)$$

The main idea

- We say a resource theory defined by the non-resource states *M* has the *exponential distinguishing* (**ED**) property if for every resource state ρ

$$\beta_n(\rho, \varepsilon) := \min_{0 \leq A_n \leq I} \left(\beta_2(A_n) : \beta_1(A_n) \leq \varepsilon \right) \approx 2^{-nE(\rho)}$$

For a non-identically zero function *E*

$$\beta_2(A_n) := \max_{\omega_n \in M} tr(\omega_n A_n) \qquad \beta_1(A_n) := tr(\rho^{\otimes n}(I - A_n))$$

Theorem I

- **Theorem:** If M satisfies **ED**, then

$$E(\rho) = \min_{\{\rho_n\}} \lim_{n\to\infty} \frac{\log(1 + r_M(\rho_n))}{n} \quad : \quad \| \rho_n - \rho^{\otimes n} \|_1 \to 0$$

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and for every σ such that $E(\sigma) > 0$

$$R(\rho \to \sigma) = E(\rho) / E(\sigma)$$

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and for every σ such that $E(\sigma) > 0$

$$R(\rho \to \sigma) = E(\rho) / E(\sigma)$$

Proof main idea: Take maps of the form

$$\Lambda_n(.) = tr(A_n .)\sigma^{\otimes n E(\sigma)/E(\rho)} + tr((I - A_n).)\pi_n$$

Theorem I

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and for every σ such that $E(\sigma) > 0$

$$R(\rho \to \sigma) = E(\rho) / E(\sigma)$$

The theorem is completely general.

The trouble is of course how to prove that the set of non-resource of interest satisfies **ED**... Difficult in general!

Theorem II

- **Theorem:** If *M* satisfies

1. Closed and convex and contain the max. mixed state
2. If $\sigma \in M_n,\ \pi \in M_m \Rightarrow \sigma \otimes \pi \in M_{n+m}$
3. If $\sigma \in M_{n+1} \Rightarrow tr_{n+1}(\sigma) \in M_n$
4. If $\sigma \in M_n \Rightarrow P_\pi \sigma P_\pi \in M_n, \forall \pi \in S_n$

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4. If $\sigma \in M_n \Rightarrow P_\pi \sigma P_\pi \in M_n, \forall \pi \in S_n$

$$P_\pi |\psi_1\rangle|\psi_2\rangle \ldots |\psi_n\rangle = |\psi_{\pi^{-1}(1)}\rangle|\psi_{\pi^{-1}(2)}\rangle \ldots |\psi_{\pi^{-1}(n)}\rangle$$

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Then **ED** holds true, $E(\rho) = E_M^\infty(\rho)$, and by the Theorem I

$$R(\rho \rightarrow \sigma) = E_M^\infty(\sigma) / E_M^\infty(\rho)$$

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Proof: Original quantum Stein's Lemma + exponential de Finetti theorem + Lagrange duality

Resource Theories

• The theorem applies to many resource theories (purity, non-classicality, etc). A notable example where it fails is in resource theories of superselection rules:

In this case, the theories do not satisfy ED (Gour, Marvian, Spekkens 09)

• There is also a notable example where the formalism works well: ***Non-locality***

Non Locality

- Alice and Bob chooses uniformly at random from *m* measurements, each with *r* outcomes, sampling from the joint distribution $p(a,b|x,y)$

- Classically, all the possible distributions have the form

$$p(a,b|x,y) = \sum_\lambda p(\lambda)p(a|x,\lambda)q(b|y,\lambda)$$

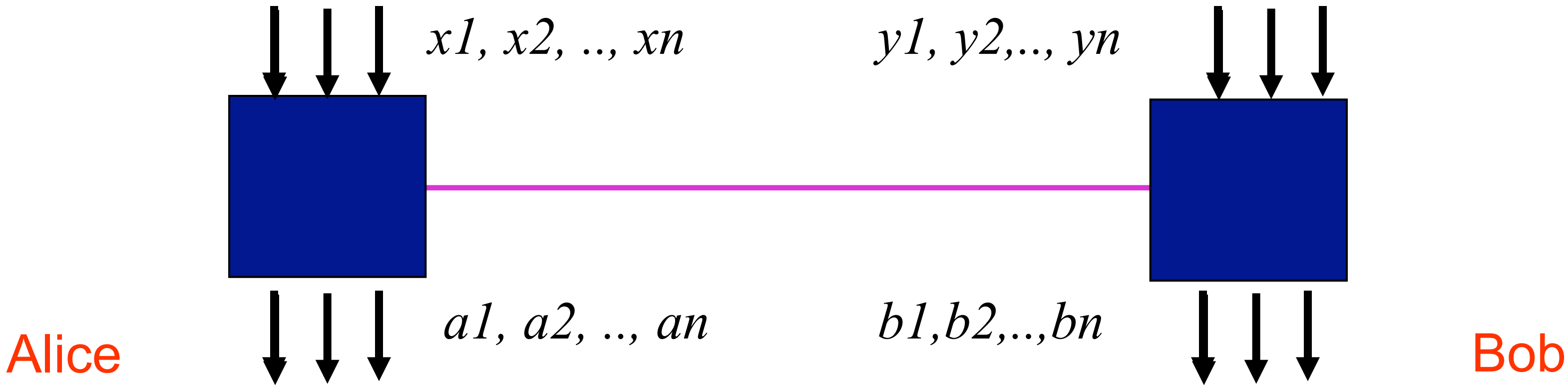
i.e. they all admit a local hidden variable theory

Non Locality

- Quantum mechanics allow more general distributions! Bell inequalities provide an way to experimentally probe this.

- Our result on hypothesis testing gives a statistically stronger “Bell test”.

Non-Locality Hypothesis Testing



- Given a sequence of random variables $\omega_n = ((x_1, a_1, y_1, b_1), \ldots, (x_n, a_n, y_n, b_n))$ with the promise that either they are distributed according to $p(a,b|x,y)$ or they are distributed according to classical distributions (admitting a LHV), but which might be arbitrary and vary for each *n*

Let $\beta_1(A_n), \beta_2(A_n)$ be the probabilities of error for the test A_n ,

Non-Locality Hypothesis Testing

- Then we have:

$$\beta_n(p,\varepsilon) := \min_{0 \le A_n \le 1} \Big(\beta_2(A_n) : \beta_1(A_n) \le \varepsilon \Big) \approx 2^{-nE_{NL}^{\infty}(p)}$$

with

$$E_{NL}(p(a,b\,|\,x,y)) := \min_{q \in LHV} S(p\,\|\,q)$$

Non-Locality *Reversibility*

- The class of operations here as just stochastic matrices mapping LHV distributions to LHV distributions. So reversibility means

- All non-local probability distributions are qualitatively the same:

n samples of an arbitrary distribution p(a, b | x, y) has the same statistical strength as $nE_{NL}^{\infty}(p)/E_{NL}^{\infty}(p_{PR})$ samples of a Popescu-Rohrlich box or $nE_{NL}^{\infty}(p)/E_{NL}^{\infty}(q)$ samples of any other distribution q!

- E_{NL} is therefore a good measure of non-locality!

Conclusions

- We have seen that it is possible to formulate a (fantasy-land) *thermodynamical* theory of entanglement and other resource theories

- This *relaxed* theory turns out to be useful for the standard paradigm of entanglement theory

- The connection of distinguishability and reversibility, at the heart of the proof of our result, has also implications to non-locality tests

Thank you!!