Physics 125a – Midterm Solutions – Due November 2, 2016

Problem 1 - Measurement in Quantum Mechanics

A quantum mechanical observable is represented by the Hermitian operator $\mathcal{O}$. It has an orthonormal basis of eigenstates $|\omega_i\rangle$, where $\mathcal{O}|\omega_i\rangle = \omega_i|\omega_i\rangle$. Suppose a system is prepared in a state $|\psi\rangle$ that is a linear combination of eigenstates of $\mathcal{O}$ with eigenvalues $\omega_i \in \{0, \pm 1\}$

$$|\psi\rangle = \frac{\alpha|+1\rangle + \beta|-1\rangle + \gamma|0\rangle}{\sqrt{|\alpha|^2 + |\beta|^2 + |\gamma|^2}}\,, \tag{1}$$

where α, β and γ are complex numbers.

(a) (5pts) *Describe the possible outcomes and their probabilities if an observer measures $\mathcal{O}$ and then measures it again and multiplies the results of those two measurements.*

In order to describe the possible measurement outcomes of finding the respective eigenvalues ω_i of $\mathcal{O}$ we note that the corresponding eigenvectors $|\omega_i\rangle$ form an orthonormal basis. In particular we can define Hermitian projection operators P_i onto the subspace of eigenvalues ω_i. They satisfy the completeness relation and are idempotent.

$$\mathbb{I} = \sum_i P_i\,, \qquad P_i^2 = P_i\,, \qquad P_iP_j = \delta_{ij}P_i \text{ (no summation on } i) \tag{2}$$

If the state before measurement is $|\psi\rangle$, then result i occurs with probability $p(i)$ given by,

$$p(i) = \langle\psi|P_i^\dagger P_i\,|\psi\rangle > 0 \tag{3}$$

and the state after measuring i, denoted by $|\psi_i\rangle$ is (correctly normalized),

$$|\psi_i\rangle = \frac{P_i|\psi\rangle}{\sqrt{\langle\psi|P_i^\dagger P_i\,|\psi\rangle}}\,. \tag{4}$$

Defining $P_{+1} = |+1\rangle\langle+1|$, $P_{-1} = |-1\rangle\langle-1|$ and $|0\rangle\langle 0|$, we obtain for the first measurement

$$|\psi_{+1}\rangle = \frac{\alpha}{|\alpha|}|+1\rangle \qquad p(+1) = \frac{|\alpha|^2}{|\alpha|^2 + |\beta|^2 + |\gamma|^2} \tag{5}$$

$$|\psi_{-1}\rangle = \frac{\beta}{|\beta|}|-1\rangle \qquad p(-1) = \frac{|\beta|^2}{|\alpha|^2 + |\beta|^2 + |\gamma|^2} \tag{6}$$

$$|\psi_0\rangle = \frac{\gamma}{|\gamma|}|0\rangle \qquad p(0) = \frac{|\gamma|^2}{|\alpha|^2 + |\beta|^2 + |\gamma|^2} \tag{7}$$

After the first measurement is done, the state collapses into an eigenstate of $\mathcal{O}$, so that we will measure the same eigenstate with probability 1 in the second measurement as the one in the first measurement and the complement with probability 0. For the states after measurements (i, j), we find

$$|\psi_{ij}\rangle = \frac{|j\rangle\langle j|\psi_i\rangle}{||j\rangle\langle j|\psi_i\rangle|^2} \tag{8}$$

with the probabilities (all other combinations are zero),

$$p(+1,+1) = \frac{|\alpha|^2}{|\alpha|^2+|\beta|^2+|\gamma|^2}\,, \quad p(-1,-1) = \frac{|\beta|^2}{|\alpha|^2+|\beta|^2+|\gamma|^2}\,, \quad p(0,0) = \frac{|\gamma|^2}{|\alpha|^2+|\beta|^2+|\gamma|^2}$$

The outcome of the product of measurements occurs with probabilities $P(i)$,

$$P(1) = \frac{|\alpha|^2+|\beta|^2}{|\alpha|^2+|\beta|^2+|\gamma|^2}\,, \qquad P(0) = \frac{|\gamma|^2}{|\alpha|^2+|\beta|^2+|\gamma|^2} \tag{9}$$

(b) (5pts) *Describe the possible outcomes and their probabilities if an observer measures* $\mathcal{O}^2$

In principle this problem is pretty straight forward, one can consider $\mathcal{O}$ written in an eigenbasis

$$\mathcal{O} = \sum_i \omega_i |\omega_i\rangle\langle\omega_i| \quad \Rightarrow \quad \mathcal{O}^2 = \sum_i \omega_i^2 |\omega_i\rangle\langle\omega_i| \tag{10}$$

and the possible measurement outcomes of $\mathcal{O}^2$ are $+1$ and 0 associated to the states $|+\rangle$ and $|0\rangle$. Note that the eigenvalue $+1$ is now degenerate and there are two states $|+1\rangle$ and $|-1\rangle$ that give the same measurement result when $\mathcal{O}^2$ is measured. To take this into account we define new projection operators for the measurement,

$$P_+ = |+1\rangle\langle +1| + |-1\rangle\langle -1|\,, \qquad P_0 = |0\rangle\langle 0|\,. \tag{11}$$

Following the same steps as in (a), we find

$$|+\rangle = \frac{\alpha|+1\rangle + \beta|-1\rangle}{\sqrt{|\alpha|^2+|\beta|^2}}\,, \qquad p(+) = \frac{|\alpha|^2+|\beta|^2}{|\alpha|^2+|\beta|^2+|\gamma|^2} \tag{12}$$

$$|0\rangle = \frac{\gamma}{|\gamma|}|0\rangle \qquad p(0) = \frac{|\gamma|^2}{|\alpha|^2+|\beta|^2+|\gamma|^2} \tag{13}$$

Note: In this problem give both the state of the system (after each measurement is made) for each value of the observable obtained from the measurement and the probability of obtaining that outcome.

Problem 2

Solution:

(a) Schrödinger's equation is

$$H\psi(t) = i\hbar\partial_t\psi(t). \tag{14}$$

Matrix-multiplying out the LHS this splits into two coupled ODEs,

$$gt\psi_2(t) = i\hbar\psi_1'(t), \tag{15}$$
$$gt\psi_1(t) = i\hbar\psi_2'(t). \tag{16}$$

There are many ways of solving this system, including writing it back as a matrix equation and diagonalizing. However, we will use the hint and take sum and difference of the equations, which decouples the system. We have

$$gt\,(\psi_1 + \psi_2) = i\hbar\left(\psi_1' + \psi_2'\right), \tag{17}$$
$$gt\,(\psi_1 - \psi_2) = -i\hbar\left(\psi_1' - \psi_2'\right). \tag{18}$$

These are two decoupled ODEs. Defining

$$\psi_S(t) \equiv \psi_1(t) + \psi_2(t), \tag{19}$$
$$\psi_D(t) \equiv \psi_1(t) - \psi_2(t). \tag{20}$$

we can solve each differential equation as

$$\psi_S(t) = (\psi_1(0) + \psi_2(0))\, e^{-\frac{igt^2}{2\hbar}}, \tag{21}$$
$$\psi_D(t) = (\psi_1(0) - \psi_2(0))\, e^{+\frac{igt^2}{2\hbar}}. \tag{22}$$

Solving back for $\psi_{1,2}(t)$ in terms of $\psi_{S,D}(t)$ gives

$$\psi_1(t) = \frac{1}{2}(\psi_1(0) + \psi_2(0))\, e^{-\frac{igt^2}{2\hbar}} + \frac{1}{2}(\psi_1(0) - \psi_2(0))\, e^{+\frac{igt^2}{2\hbar}}, \tag{23}$$
$$\psi_2(t) = \frac{1}{2}(\psi_1(0) + \psi_2(0))\, e^{-\frac{igt^2}{2\hbar}} - \frac{1}{2}(\psi_1(0) - \psi_2(0))\, e^{+\frac{igt^2}{2\hbar}}. \tag{24}$$

The above can be expressed more naturally in terms of sin and cos,

$$\psi_1(t) = \psi_1(0)\cos\frac{gt^2}{2\hbar} - i\psi_2(0)\sin\frac{gt^2}{2\hbar}, \tag{25}$$
$$\psi_2(t) = \psi_2(0)\cos\frac{gt^2}{2\hbar} - i\psi_1(0)\sin\frac{gt^2}{2\hbar}. \tag{26}$$

(b) For the given initial conditions we have

$$\psi_1(t) = \cos\frac{gt^2}{2\hbar}, \tag{27}$$
$$\psi_2(t) = -i\sin\frac{gt^2}{2\hbar}, \tag{28}$$

so the probability is simply

$$P(t) = \sin^2\frac{gt^2}{2\hbar}. \tag{29}$$

(a) Express Schrödinger's equation for the time evolution of $|\Psi(t)\rangle$ as first order coupled differential equations for the components $\psi_1(t)$ and $\psi_2(t)$. Taking linear combinations of the two equations convert them into uncoupled first order differential equations. Solve them and express $\psi_1(t)$ and $\psi_2(t)$ in terms of their initial values $\psi_1(0)$ and $\psi_2(0)$.

(b) Derive an expression for the probability that a particle inititially in the state with $\psi_1 = 1$, $\psi_2 = 0$ is in the state with $\psi_1 = 0$, $\psi_2 = 1$ after a time t.

Problem 3

We consider a nonrelativistic particle of mass m moving in one-dimension in a gravitational potential, $V = mgz$. We want to compute the propagator from $(0, 0)$ to (z_2, t_2)

$$\langle z_2|U(t_2, 0)|0\rangle \simeq e^{iS[z(t)]/\hbar}\,, \tag{30}$$

where we find the propagator, in the stationary phase approximation of the path integral, by evaluating the action on its classical path. We compute the equation of motion from the action as

$$S = \int_{t_i}^{t_f} dt \left(\frac{1}{2}m\dot{z}(t)^2 - mgz(t)\right) \quad \to \quad \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{z}} - \frac{\partial \mathcal{L}}{\partial z} = m\ddot{z} + mg = 0\,. \tag{31}$$

Integrating the equation of motion $\ddot{z}(t) = -g$, to find $z(t)$

$$\dot{z}(t) = v_0 - gt\,, \qquad z(t) = z_0 + v_0 t - \frac{1}{2}gt^2\,, \tag{32}$$

where the boundary conditions for this path, that $z(0) = 0$ and $z(t_2) = z_2$, require us to fix

$$z_0 = 0\,, \qquad v_0 = \frac{z_2}{t_2} + \frac{1}{2}gt_2\,. \tag{33}$$

Evaluating the action on the classical path $z(t)$, we find

$$\int_0^{t_2} dt\, m\left(\frac{1}{2}v_0^2 + g^2t^2 - 2gv_0 t\right) = m\left(\frac{1}{2}v_0^2 t_2 + \frac{1}{3}g^2 t_2^3 - gv_0 t_2^2\right). \tag{34}$$

Using v_0 above, we find

$$\langle z_2|U(t_2, 0)|0\rangle \simeq \exp\left(\frac{im}{\hbar}\left(\frac{1}{2t_2}z_2^2 - \frac{gt_2}{2}z_2 - \frac{g^2t_2^3}{24}\right)\right)\,, \tag{35}$$

and thus $A = \frac{1}{2t_2}$, $B = -\frac{1}{2}gt_2$, and $C = -\frac{1}{24}g^2t_2^3$.