Is quantum computation helpful for approximating partition functions?

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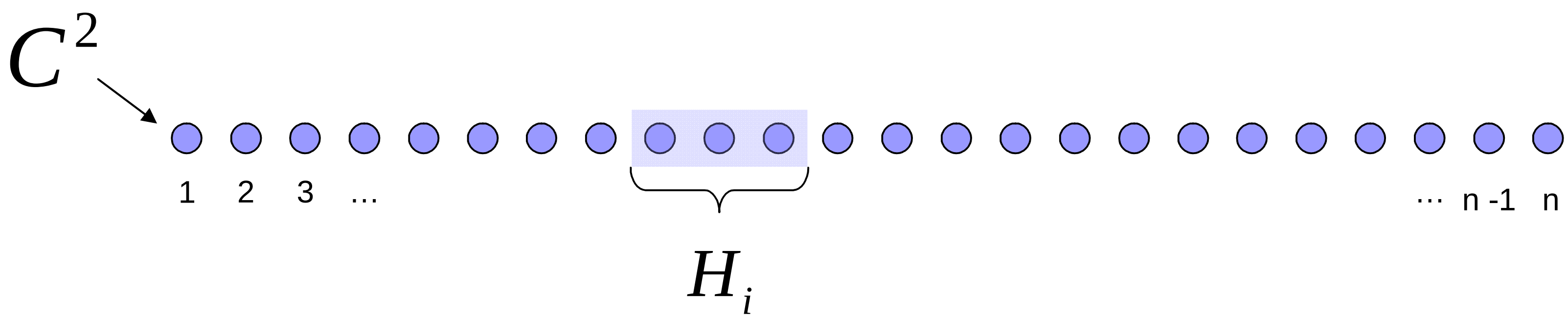
Madrid, 28/05/2008

Imperial College London

Institute for Mathematical Sciences

QIIC

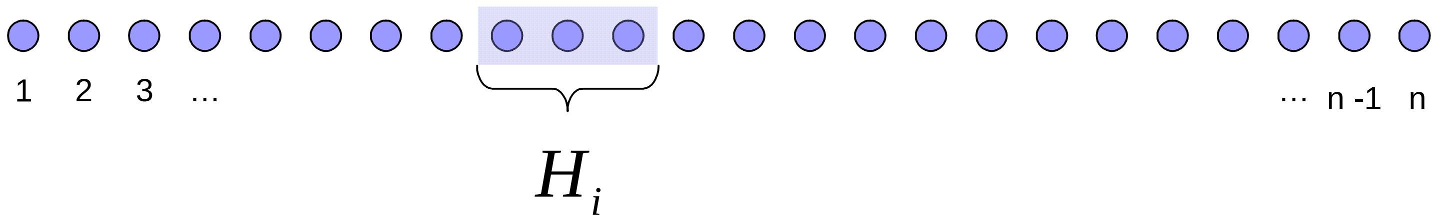
Partition Function



- Local Hamiltonian: $H = \sum_{k=1}^{POLY(n)} H_i \qquad \| H_i \| = O(1)$
- Partition function: $$Z(H,\beta) = tr(e^{-\beta H}) = \sum_k e^{-\beta \lambda_k (H)}$$

Partition Function

$$H = \sum_{k=1}^{POLY(n)} H_i$$



- How hard it is to calculate partition functions ?

Partition Function

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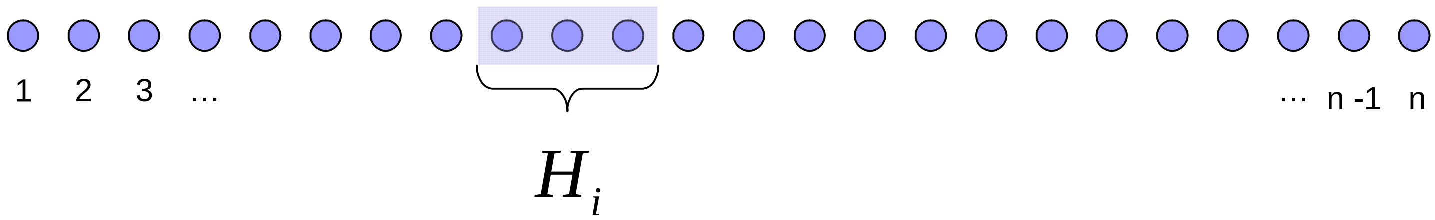

- How hard it is to calculate partition functions ?
- Local classical Hamiltonians: #P-hard
- How about approximations?

Partition Function

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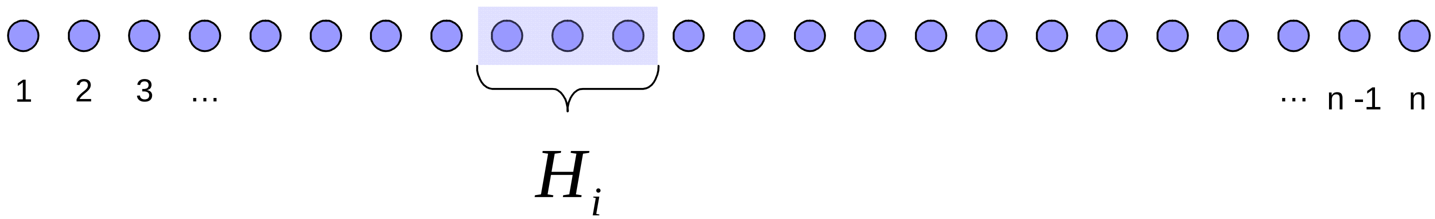


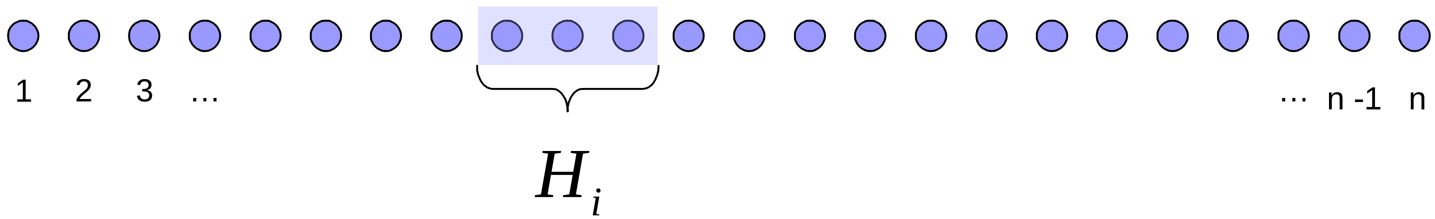
- Multiplicative approximation: Find a χ such that with with high probability

$$\left|Z(H,\beta) - \chi\right| \leq Z(H,\beta)\varepsilon$$

- Efficiency: should be polynomial in *n* and ε

Partition Function

$$H = \sum_{k=1}^{POLY(n)} H_i$$



- Multiplicative approximation: Find a χ such that with high probability

$$\left|Z(H,\beta) - \chi\right| \leq Z(H,\beta)\varepsilon$$

- NP-hard even for classical local Hamiltonians…

- Intractable both for classical and quantum computation

Unless $NP \subseteq BQP$

Partition Function

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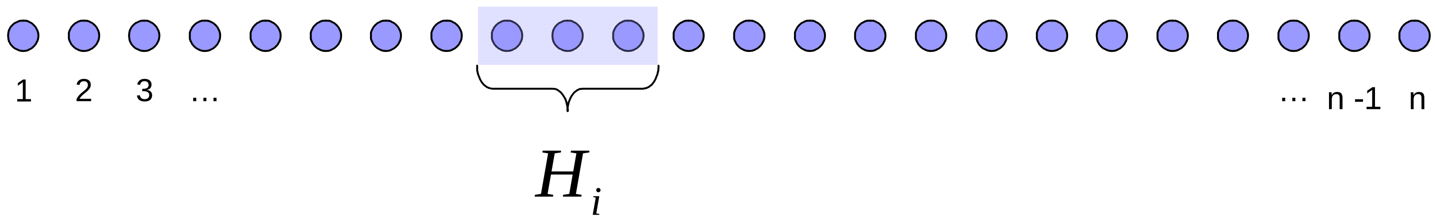

- *Additive* approximation: Find a χ such that with high probability

$$\left| Z(H,\beta) - \chi \right| \leq \Delta\varepsilon$$

- $\Delta(n)$: additive window of the approximation
- For Δ too large the approximation is trivial…

Partition Function

Is a quantum computer helpful for approximating partition functions?

- Old question, no substantial progress for a long time

Lidar&Biham, PRE 97 Lidar NJP 04

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- Recently, intriguing polynomial quantum algorithms for additive approx. of classical partition functions:

Aharonov, Arad, Eban, Landau 07

Arad&Landau 08

Bravyi&Raussendorf PRA 2007

Van den Nest, Dür, Briegel PRL 07, PRL 08

Bombin&Martin-Delgado 07

} Implicit quantum algorithms

Van den Nest, Dür, Raussendorf, Briegel 08

Partition Function

Is a quantum computer helpful for approximating partition functions?

- Are these algorithms non-trivial? Can we compute the same approximation on a classical computer?

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Is a quantum computer helpful for approximating partition functions?

- Are these algorithms non-trivial? Can we compute the same approximation on a classical computer?

  - For Hamiltonians with *complex* couplings: **BQP-complete** approximations!

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Partition Function

Is a quantum computer helpful for approximating partition functions?

- Are these algorithms non-trivial? I.e. can we compute the same approximation on a classical computer?
  - For classical Hamiltonians with *complex* couplings: **BQP-complete** approximations!
    - How about for physical parameters?

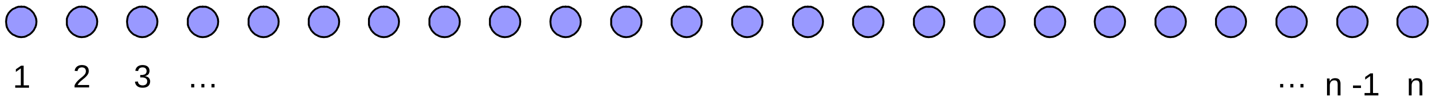
Next 10 Minutes

1. Quantum algorithm for additive approximations of *quantum* local Hamiltonians

2. Non-triviality proof: problem solved is *complete* for the one-clean-qubit model of quantum computation (DQC1)

3. DQC1-complete algorithms for estimating the *spectrum density* of local quantum Hamiltonians (time permitting)

Classical Algorithm for additive approximation of classical Hamiltonian partition functions



- For classical Hamiltonians it is easy to sample from $\{\lambda_k(H)\}$ with uniform probability

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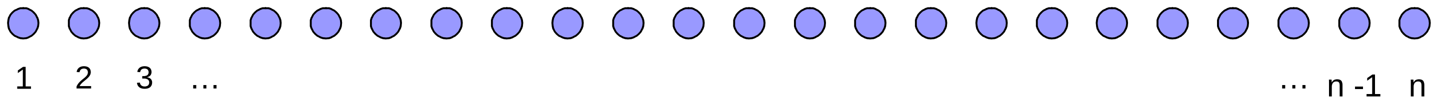

- For classical Hamiltonians it is easy to sample from $\{\lambda_k(H)\}$ with uniform probability
- Pick $\lambda^1, \lambda^2, ..., \lambda^r$ at random from $\{\lambda_k(H)\}$ and compute

$$\chi_r = \frac{1}{r}\sum_{k=1}^{r} e^{-\beta\lambda^k}$$

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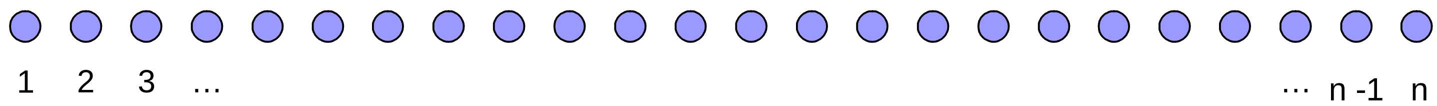

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$$\chi_r = \frac{1}{r}\sum_{k=1}^{r} e^{-\beta\lambda^k} \qquad \overline{\chi_r} = \frac{1}{2^n}\sum_{k=1}^{2^n} e^{-\beta\lambda_k(H)} = \frac{Z(H,\beta)}{Z(H,\infty)}$$

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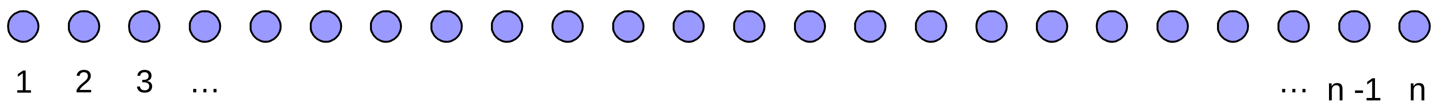

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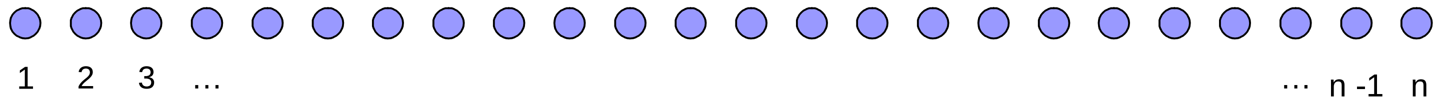
- By Hoeffding's inequality

$$\Pr\left(\left|\chi_r - \frac{Z(H,\beta)}{Z(H,\infty)}\right| \geq \varepsilon\right) \leq e^{-r\varepsilon^2}$$

Quantum Algorithm for additive approximation of quantum Hamiltonian partition functions



- For quantum Hamiltonians it is **not** easy to sample from $\{\lambda_k(H)\}$ with uniform probability *in a classical computer*

- But in a quantum computer we can do it. **Phase estimation algorithm:**

(*Input*) unitary U with polynomially many one- and two-qubit gates and a $|u\rangle$ s.t. $U|u\rangle = e^{i\pi\theta}|u\rangle$

(*Output*) an approximation to θ with accuracy 1/POLY(n)

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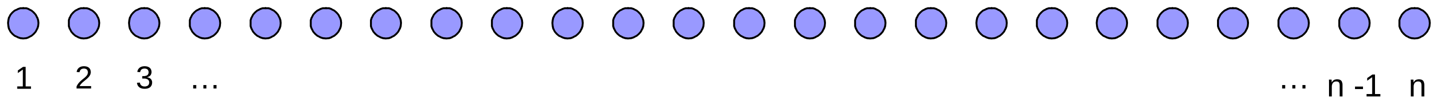

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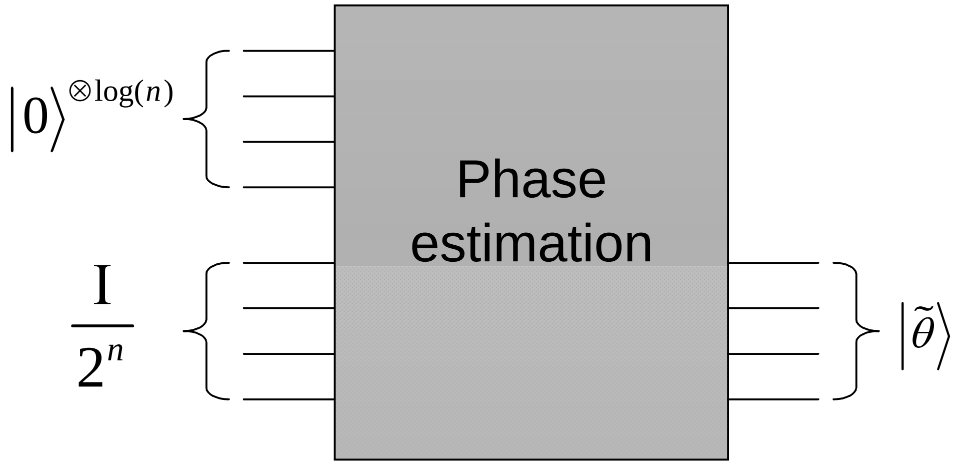
- Choose $U = e^{iH}$ and input $\frac{I}{2^n}$ instead of $|u\rangle$

Quantum Algorithm for additive approximation of quantum Hamiltonian partition functions

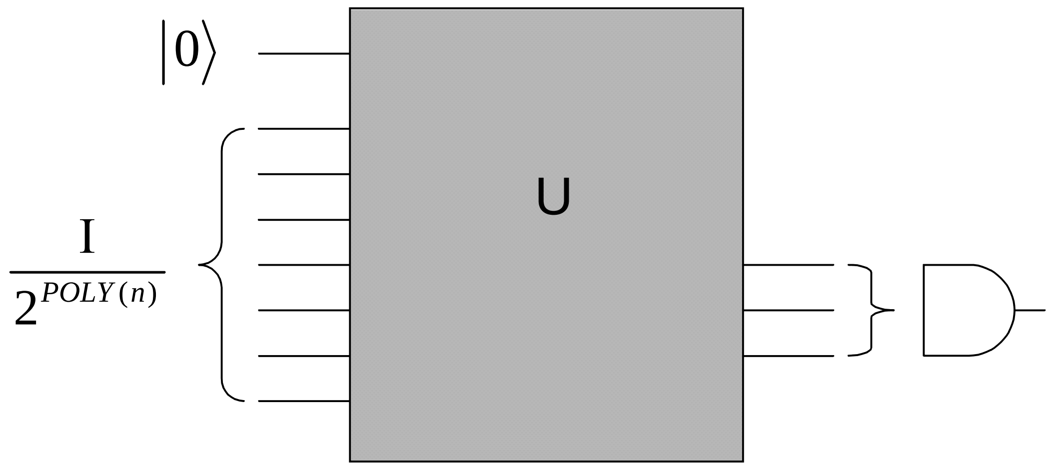


Only require log(n) *clean* qubits: The problem is in **DQC1** (one-clean-qubit model of quantum computation)

Knill & Laflamme PRL 98

Shor & Jordan QIC 08

DQC1(one-clean-qubit model)

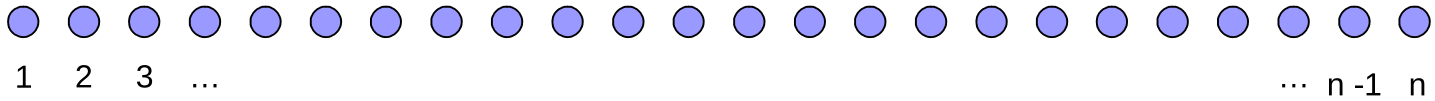


- DQC1 is believed to lie in between P and BQP

- Complete problems:

Additive approximation for the trace of quantum circuits	Knill & Laflamme PRL 98
Additive approximation for (some) quadratically signed weight enumerators	Knill & Laflamme PRL 99
Additive approximation for the Jones polynomial of the plat closure of a braid	Aharonov, Jones, Landau STOC 06 Shor & Jordan QIC 08

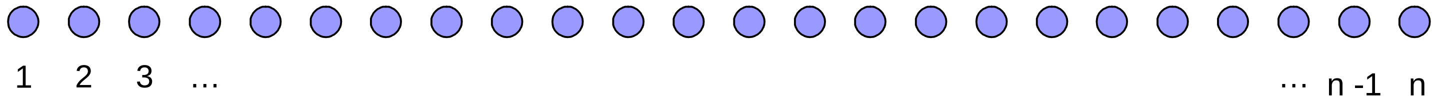
DQC1-completeness



Teo: To calculate additive approximations of partition functions of local quantum Hamiltonians with an approx. window $\Delta = Z(H, \infty)$ is a complete problem for DQC1

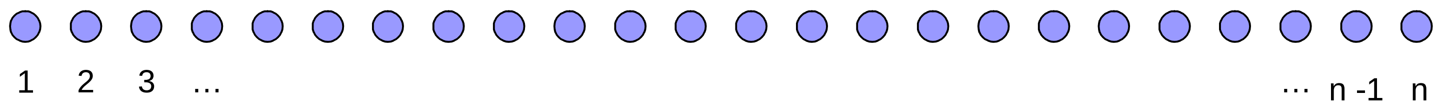
- Therefore, unless $DQC1 \subseteq P$, quantum computation is helpful for approximating quantum partition functions!

DQC1-hardness



Proof main idea: encode solution of a DQC1-problem into the low-lying sprectrum of a local Hamiltonian, using Kitaev's construction of maping the solution of a quantum circuit into properties of local quantum Hamiltonians

DQC1-hardness



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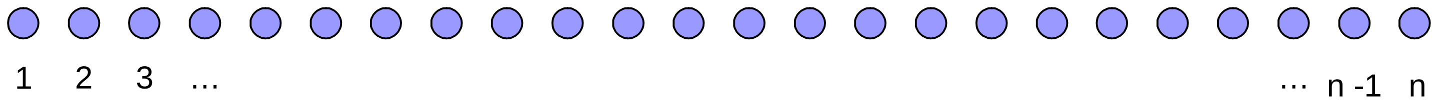
QMA : ground state energy Kitaev 00

BQP : local observable of adiabatically connected Hamiltonian Aharonov *et al* FOCS 04

$NP \cap co-NP$: ground state energy of poly-gapped Hamiltonian with a MPS ground state Schuch, Cirac, Verstraete 08

DQC1-hardness



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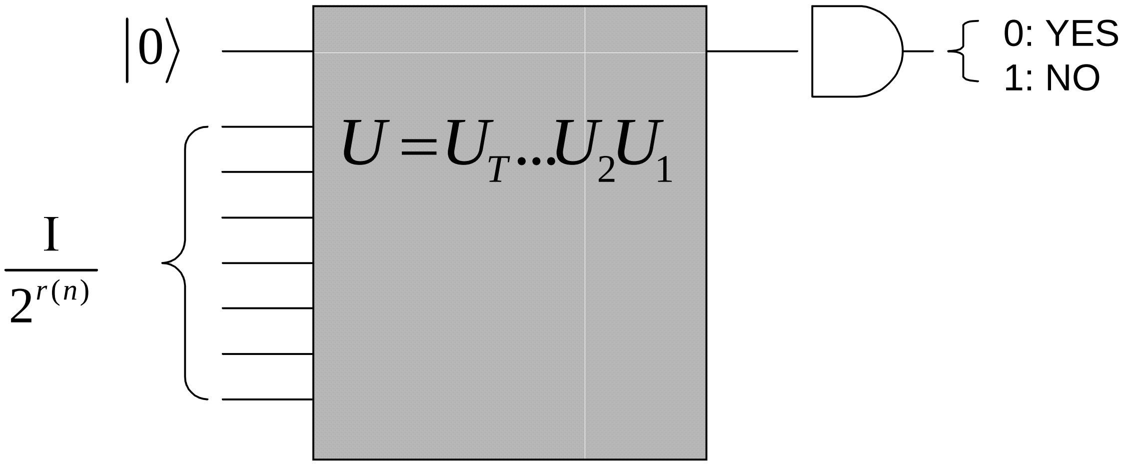
DQC1: average energy of the low-lying energy spectrum of a poly-gapped Hamiltonian (degeneracy of the ground space)

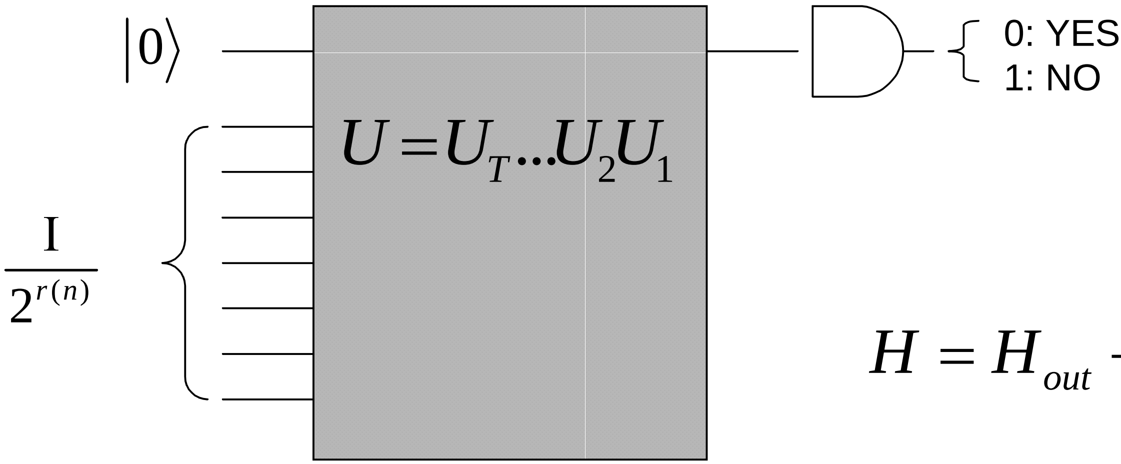
DQC1-hardness



$$r(n) = POLY(n)$$

$$T(n) = POLY(n)$$

DQC1-hardness

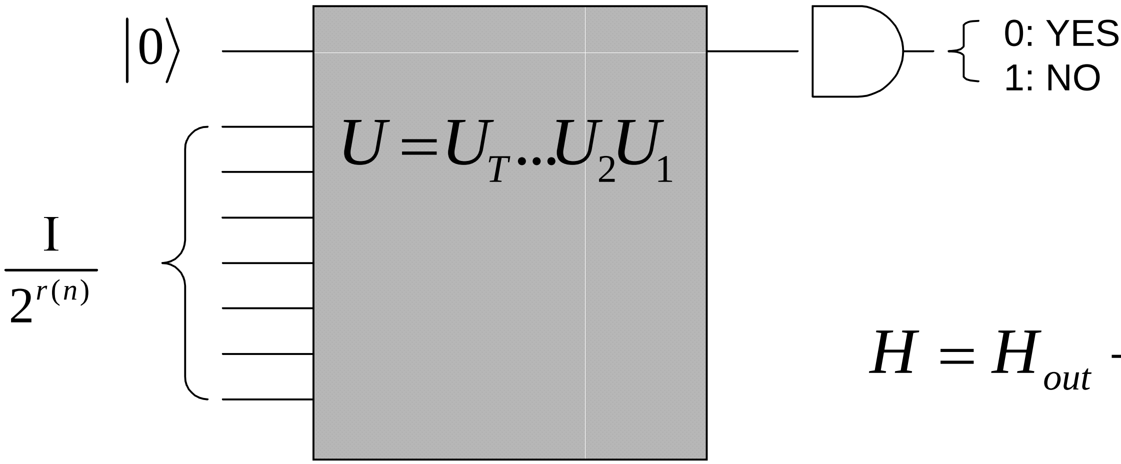


$$r(n) = POLY(n)$$

$$T(n) = POLY(n)$$

$$H = H_{out} + J_{in} H_{in} + J_{ptop} H_{prop}$$

DQC1-hardness



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$$H_{in} = |1\rangle\langle 1|_1 \otimes |0\rangle\langle 0|_{clock}$$

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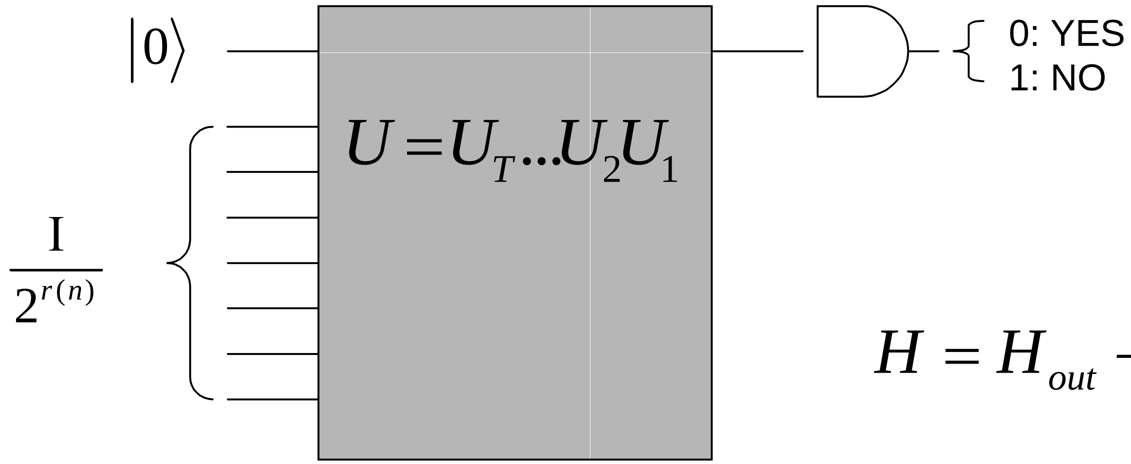
DQC1-hardness



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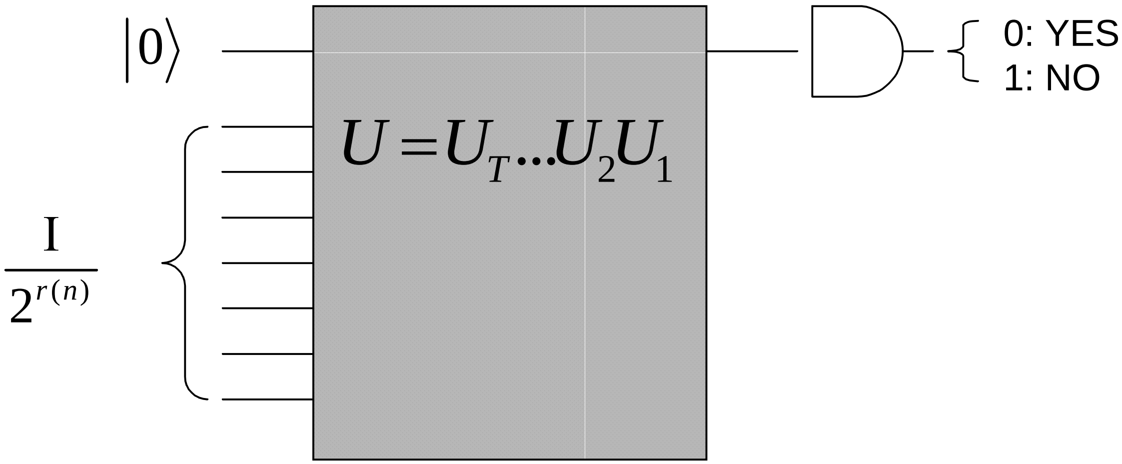
$$H_{prop} = \sum_{i=1}^{T} H_{prop,i}$$

$$H_{prop,t} = I \otimes |t\rangle\langle t|_{clock} + I \otimes |t-1\rangle\langle t-1|_{clock} - U_t^{\dagger} \otimes |t-1\rangle\langle t|_{clock} - U_t \otimes |t\rangle\langle t-1|_{clock}$$

DQC1-hardness



$$r(n) = POLY(n)$$
$$T(n) = POLY(n)$$

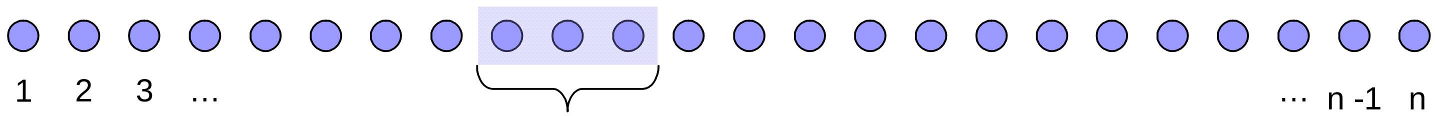
If YES: $2^{r(n)}$ eigenvalues are almost zero

If NO: at most $\varepsilon 2^{r(n)}$ eigenvalues are smaller than δ

$$\frac{Z(H,\beta)}{Z(H,\infty)} = \frac{1}{2^{r(n)+\log(T(n))+1}} \sum_k e^{-\beta\lambda_k(H)}$$

Spectrum Density

$$H = \sum_{k=1}^{POLY(n)} H_i$$



$$H_i$$

Eigenvalue density

$$\mu_H(x) = \frac{1}{2^n} \sum_{j=1}^{2^n} \delta(\lambda_j(H) - x)$$

Counting function

$$N_H(a,b) = \int_a^b \mu_H(x) dx$$

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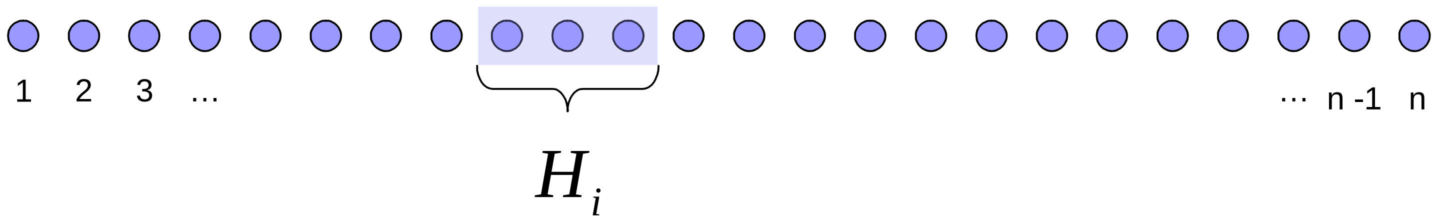

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> To approximate the counting function of local quantum Hamiltonians to accuracy $\varepsilon = 1/POLY(n)$ is a DQC1-complete problem

Interestingly, there is classical algorithm for the problem polynomial in *n* but exponential in ε

Osborne 06

Thank you!