Is quantum computation helpful for approximating partition functions?

Fernando G.S.L. Brandão and Martin B. Plenio

Madrid, 28/05/2008
Local Hamiltonian: \[ H = \sum_{k=1}^{\text{POLY}(n)} H_i \quad \| H_i \| = O(1) \]

Partition function: \[ Z(H, \beta) = \text{tr}(e^{-\beta H}) = \sum_{k} e^{-\beta \lambda_k(H)} \]
How hard it is to calculate partition functions?
How hard it is to calculate partition functions?

- Local classical Hamiltonians: #P-hard

- How about approximations?
Partition Function

- Multiplicative approximation: Find a $\chi$ such that with high probability

$$|Z(H, \beta) - \chi| \leq Z(H, \beta) \varepsilon$$

- Efficiency: should be polynomial in $n$ and $\varepsilon$
Partition Function

\[ H = \sum_{k=1}^{\text{POLY}(n)} H_i \]

- Multiplicative approximation: Find a \( \chi \) such that with high probability

\[ |Z(H, \beta) - \chi| \leq Z(H, \beta) \varepsilon \]

- NP-hard even for classical local Hamiltonians…

- Intractable both for classical and quantum computation

  Unless \( NP \subseteq BQP \)
Additive approximation: Find a $\chi$ such that with high probability

$$\left| Z(H, \beta) - \chi \right| \leq \Delta \varepsilon$$

- $\Delta(n)$: additive window of the approximation

For $\Delta$ too large the approximation is trivial...
Is a quantum computer helpful for approximating partition functions?

- Old question, no substantial progress for a long time

Lidar & Biham, PRE 97  
Lidar, NJP 04  
Aharonov, Arad, Eban, Landau, 2007  
Arad & Landau, 2008  
Bravyi & Raussendorf, PRA 2007  
Van den Nest, Dür, Briegel, PRL 2007, PRL 2008  
Bombin & Martin-Delgado, 2007  
Van den Nest, Dür, Raussendorf, Briegel, 2008
Is a quantum computer helpful for approximating partition functions?

- Old question, no substantial progress for a long time
  
  Lidar&Biham, PRE 97        Lidar NJP 04

- Recently, intriguing polynomial quantum algorithms for additive approx. of classical partition functions:
  
  Aharonov, Arad, Eban, Landau 07
  Arad&Landau 08
  
  Bravyi&Raussendorf PRA 2007
  Van den Nest, Dür, Briegel PRL 07, PRL 08
  Bombin&Martin-Delgado 07

  Van den Nest, Dür, Raussendorf, Briegel 08
Partition Function

Is a quantum computer helpful for approximating partition functions?

- Are these algorithms non-trivial? Can we compute the same approximation on a classical computer?

Aharonov, Arad, Eban, Landau 07
Arad&Landau 08

Bravyi&Raussendorf PRA 2007
Van den Nest, Dür, Briegel PRL 07, PRL 08
Bombin&Martin-Delgado 07

Van den Nest, Dür, Raussendorf, Briegel 08
Is a quantum computer helpful for approximating partition functions?

- Are these algorithms non-trivial? Can we compute the same approximation on a classical computer?

- For Hamiltonians with complex couplings: **BQP-complete** approximations!

**References**

Aharonov, Arad, Eban, Landau 07
Arad&Landau 08

Bravyi&Raussendorf PRA 2007
Van den Nest, Dür, Briegel PRL 07, PRL 08
Bombin&Martin-Delgado 07

Van den Nest, Dür, Raussendorf, Briegel 08
Is a quantum computer helpful for approximating partition functions?

- Are these algorithms non-trivial? I.e. can we compute the same approximation on a classical computer?

- For classical Hamiltonians with complex couplings: BQP-complete approximations!

- How about for physical parameters?
Next 10 Minutes

1. Quantum algorithm for additive approximations of quantum local Hamiltonians

2. Non-triviality proof: problem solved is complete for the one-clean-qubit model of quantum computation (DQC1)

3. DQC1-complete algorithms for estimating the spectrum density of local quantum Hamiltonians (time permitting)
Classical Algorithm for additive approximation of classical Hamiltonian partition functions

- For classical Hamiltonians it is easy to sample from $\{\lambda_k(H)\}$ with uniform probability.
Classical Algorithm for additive approximation of classical Hamiltonian partition functions

• For classical Hamiltonians it is easy to sample from \( \{ \lambda_k(H) \} \) with uniform probability

• Pick \( \lambda^1, \lambda^2, \ldots, \lambda^r \) at random from \( \{ \lambda_k(H) \} \) and compute

\[
\chi_r = \frac{1}{r} \sum_{k=1}^{r} e^{-\beta \lambda^k}
\]
Classical Algorithm for additive approximation of classical Hamiltonian partition functions

- For classical Hamiltonians it is easy to sample from \( \{ \lambda_k(H) \} \) with uniform probability.
- Pick \( \lambda^1, \lambda^2, ..., \lambda^r \) at random from \( \{ \lambda_k(H) \} \) and compute

\[
\chi_r = \frac{1}{r} \sum_{k=1}^{r} e^{-\beta \lambda^k}
\]

\[
\chi_r = \frac{1}{2^n} \sum_{k=1}^{2^n} e^{-\beta \lambda_k(H)} = \frac{Z(H, \beta)}{Z(H, \infty)}
\]
Classical Algorithm for additive approximation of classical Hamiltonian partition functions

- For classical Hamiltonians it is easy to sample from \( \{ \lambda_k(H) \} \) with uniform probability.
- Pick \( \lambda^1, \lambda^2, \ldots, \lambda^r \) at random from \( \{ \lambda_k(H) \} \) and compute

\[
\chi_r = \frac{1}{r} \sum_{k=1}^{r} e^{-\beta \lambda^k}
\]

\[
\tilde{\chi}_r = \frac{1}{2^n} \sum_{k=1}^{2^n} e^{-\beta \lambda^k(H)} = \frac{Z(H, \beta)}{Z(H, \infty)}
\]

- By Hoeffding’s inequality

\[
\Pr\left( \left| \chi_r - \frac{Z(H, \beta)}{Z(H, \infty)} \right| \geq \varepsilon \right) \leq e^{-r \varepsilon^2}
\]
Quantum Algorithm for additive approximation of quantum Hamiltonian partition functions

- For quantum Hamiltonians it is not easy to sample from \( \{ \lambda_k (H) \} \) with uniform probability in a classical computer.

- But in a quantum computer we can do it. **Phase estimation algorithm:**

  (Input) unitary \( U \) with polynomially many one- and two-qubit gates and a \( |u\rangle \) s.t. \( U |u\rangle = e^{i \pi \theta} |u\rangle \)

  (Output) an approximation to \( \theta \) with accuracy \( 1/\text{POLY}(n) \)
Quantum Algorithm for additive approximation of quantum Hamiltonian partition functions

- For quantum Hamiltonians it is not easy to sample from \( \lambda_k (\hat{H}) \) with uniform probability in a classical computer.

- But if a quantum computer we can do it: Phase estimation algorithm:
  
  \( \text{(Input)} \) unitary \( U \) with polynomially many one- and two-qubit gates and a \( |u\rangle \) s.t. \( U|u\rangle = e^{i\theta}|u\rangle \)

  \( \text{(Output)} \) an approximation to \( \theta \) with accuracy \( 1/\text{POLY}(n) \)

- Choose \( U = e^{iH} \) and input \( \frac{1}{2^n} \) instead of \( |u\rangle \)
Quantum Algorithm for additive approximation of quantum Hamiltonian partition functions

Only require $\log(n)$ clean qubits: The problem is in DQC1 (one-clean-qubit model of quantum computation)

Knill & Laflamme PRL 98
Shor & Jordan QIC 08
DQC1 (one-clean-qubit model)

• DQC1 is believed to lie in between P and BQP

• Complete problems:
  - Additive approximation for the trace of quantum circuits
  - Additive approximation for (some) quadratically signed weight enumerators
  - Additive approximation for the Jones polynomial of the plat closure of a braid

Additive approximation for the trace of quantum circuits
Additive approximation for (some) quadratically signed weight enumerators
Additive approximation for the Jones polynomial of the plat closure of a braid

Knill & Laflamme PRL 98
Knill & Laflamme PRL 99
Aharonov, Jones, Landau STOC 06
Shor & Jordan QIC 08
DQC1-completeness

Teo: To calculate additive approximations of partition functions of local quantum Hamiltonians with an approx. window $\Delta = Z(H, \infty)$ is a complete problem for DQC1

• Therefore, unless $DQC1 \subseteq P$, quantum computation is helpful for approximating quantum partition functions!
DQC1-hardness

Proof main idea: encode solution of a DQC1-problem into the low-lying spectrum of a local Hamiltonian, using Kitaev’s construction of mapping the solution of a quantum circuit into properties of local quantum Hamiltonians.
DQC1-hardness

Proof main idea: encode solution of a DQC1-problem into the low-lying spectrum of a local Hamiltonian, using Kitaev’s construction of mapping the solution of a quantum circuit into properties of local quantum Hamiltonians

QMA : ground state energy

BQP : local observable of adiabatically connected Hamiltonian

$NP \cap co-NP$ : ground state energy of poly-gapped Hamiltonian with a MPS ground state

Kitaev 00

Aharonov et al FOCS 04

Schuch, Cirac, Verstraete 08
DQC1-hardness

Proof main idea: encode solution of a DQC1-problem into the low-lying spectrum of a local Hamiltonian, using Kitaev’s construction of mapping the solution of a quantum circuit into properties of local quantum Hamiltonians.

QMA : ground state energy \[ \text{Kitaev 00} \]

BQP : local observable of adiabatically connected Hamiltonian \[ \text{Aharonov et al FOCS 04} \]

\[ \text{NP} \cap \text{co-NP} \] : ground state energy of poly-gapped Hamiltonian with a MPS ground state \[ \text{Schuch, Cirac, Verstraete 08} \]

DQC1: average energy of the low-lying energy spectrum of a poly-gapped Hamiltonian (degeneracy of the ground space)
DQC1-hardness

\[ U = U_T \ldots U_2 U_1 \]

\[ \frac{I}{2^{r(n)}} \]

\[ 0: \text{YES} \]
\[ 1: \text{NO} \]

\[ r(n) = \text{POLY}(n) \]
\[ T(n) = \text{POLY}(n) \]
DQC1-hardness

$r(n) = POLY(n)$

$T(n) = POLY(n)$

$H = H_{out} + J_{in}H_{in} + J_{ptop}H_{prop}$
DQC1-hardness

\[ H = H_{out} + J_{in} H_{in} + J_{ptop} H_{prop} \]

\[ H_{in} = |1\rangle\langle 1|_I \otimes |0\rangle\langle 0|_{clock} \]

\[ H_{out} = |0\rangle\langle 0|_I \otimes |T\rangle\langle T|_{clock} \]
DQC1-hardness

\[ U = U_T ... U_2 U_1 \]

\[ \frac{1}{2^{r(n)}} \]

\[ H = H_{out} + J_{in} H_{in} + J_{prop} H_{prop} \]

\[ H_{in} = |1\rangle \langle 1| \otimes |0\rangle \langle 0|_{clock} \]

\[ H_{out} = |0\rangle \langle 0| \otimes |T\rangle \langle T|_{clock} \]

\[ H_{prop} = \sum_{i=1}^{T} H_{prop,i} \]

\[ H_{prop,t} = I \otimes |t\rangle \langle t|_{clock} + I \otimes |t-1\rangle \langle t-1|_{clock} - U_t^\dagger \otimes |t-1\rangle \langle t|_{clock} - U_t \otimes |t\rangle \langle t-1|_{clock} \]

\[ r(n) = POLY(n) \]

\[ T(n) = POLY(n) \]
DQC1-hardness

If YES: $2^{r(n)}$ eigenvalues are almost zero

If NO: at most $\varepsilon 2^{r(n)}$ eigenvalues are smaller than $\delta$

$$\frac{Z(H, \beta)}{Z(H, \infty)} = \frac{1}{2^{r(n)+\log(T(n))+1}} \sum_k e^{-\beta \lambda_k(H)}$$

$r(n) = POLY(n)$

$T(n) = POLY(n)$
Spectrum Density

\[ H = \sum_{k=1}^{POLY(n)} H_i \]

\[ H_i \]

Eigenvalue density

\[ \mu_H(x) = \frac{1}{2^n} \sum_{j=1}^{2^n} \delta(\lambda_j(H) - x) \]

Counting function

\[ N_H(a, b) = \int_a^b \mu_H(x) dx \]
Spectrum Density

\[ H = \sum_{k=1}^{POLY(n)} H_i \]

To approximate the counting function of local quantum Hamiltonians to accuracy \( \varepsilon = 1/POLY(n) \) is a DQC1-complete problem.

Interestingly, there is classical algorithm for the problem polynomial in \( n \) but exponential in \( \varepsilon \)

Osborne 06
Thank you!