Entanglement Area Law (from Heat Capacity)

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Based on joint work arXiv:1410.XXXX with

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University of Ulm

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Plan

- What is an area law?
- Relevance
- Previous Work
- Area Law from Heat Capacity
Area Law

\[ |\psi\rangle \in (\mathbb{C}^2)^\otimes n \quad 4^n \text{ parameters} \]
Area Law

\[ |\psi\rangle \in (\mathbb{C}^2)^{\otimes n} \]

4^n parameters

Quantum states on a lattice

\[ \mathbb{C}^2 \]

\[ \partial R \text{ : boundary of } R \]
\[ |R| \text{ : volume of } R \]
\[ |\partial R| \text{ : volume of } \partial R \]
Area Law

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Area Law

\[ |\psi\rangle \in (\mathbb{C}^2)^\otimes n \]

4^n parameters

Quantum states on a lattice

Def: Area Law holds for \(|\psi\rangle\) if for all \(R\),

\[ S(\text{tr}_{R^c}(|\psi\rangle\langle\psi|)) \leq O(|\partial R|) \]

\(\partial R\) : boundary of \(R\)

\(|R|\) : volume of \(R\)

\(|\partial R|\) : volume of \(\partial R\)
When does area law hold?

1st guess: it holds for every low-energy state of local models

\[ H = \sum_{<i,j>} H_{i,j}, \quad \|H_{i,j}\| \leq 1 \]

Energy of \(|\psi\rangle\): \( \langle \psi | H | \psi \rangle \)

\[ H = \sum_{k} E_k |E_k\rangle\langle E_k|, \quad E_0 \leq E_1 \leq \ldots \]

\( E_0 \): ground energy

\( |E_0\rangle \): ground state
When does area law hold?

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(Irani ‘07, Gottesman&Hastings ‘07)

There are 1D models with volume scaling of entanglement in groundstate
When does area law hold?

1st guess: it holds for every low-energy state of local models (Irani ‘07, Gottesman&Hastings ‘07)
There are 1D models with volume scaling of entanglement in groundstate

Must put more restrictions on Hamiltonian/State!

- spectral gap
- Correlation length
- specific heat
1D: Area Law $S_{\alpha}, \alpha < 1$  

Rényi Entropies: 

$$S_{\alpha}(\rho) := \frac{1}{1 - \alpha} \log \text{tr}(\rho^\alpha)$$

Matrix-Product-State: 

$$|\psi\rangle = \sum_{i_1, \ldots, i_n} \text{tr}(A_{i_1} \ldots A_{i_n}) |i_1, \ldots, i_n\rangle$$

Good Classical Description (MPS)  

(FNW ’91 Vid ’04)
Relevance

1D:
Area Law
\( S_\alpha, \, \alpha < 1 \)

(appears to be connected with good tensor network description; e.g. PEPS, MERA)

Renyi Entropies:
\[
S_\alpha(\rho) := \frac{1}{1 - \alpha} \log \text{tr}(\rho^\alpha)
\]

Matrix-Product-State:
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Previous Work

(Bekenstein ‘73, Bombelli et al ‘86, ....)
Black hole entropy

(Vidal et al ‘03, Plenio et al ’05, ...)
Integrable quasi-free bosonic systems and spin systems :

see Rev. Mod. Phys. (Eisert, Cramer, Plenio ‘10)
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(Bekenstein ‘73, Bombelli et al ‘86, ....)
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2\textsuperscript{nd} guess: Area Law holds for

1. Groundstates of gapped Hamiltonians
2. Any state with finite correlation length
Gapped Models

Def:

<table>
<thead>
<tr>
<th>(gap)</th>
<th>$\Delta(H) := E_1(H) - E_0(H)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(gapped model)</td>
<td>${H_n}$ gapped if $\exists \Delta &gt; 0, \Delta(H_n) \geq \Delta \ \forall \ n$</td>
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Gapped Models

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(gapped model) \( \{H_n\} \) gapped if \( \exists \Delta > 0, \Delta(H_n) \geq \Delta \forall n \)

\[
|\langle \psi | A \otimes B | \psi \rangle - \langle \psi | A | \psi \rangle \langle \psi | B | \psi \rangle| \leq 2^{-\text{dist}(A,B)/\xi}
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Gapped Models

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- (Has '04)
- finite correlation length
- Exponential small heat capacity

\[ |\langle \psi | A \otimes B | \psi \rangle - \langle \psi | A | \psi \rangle \langle \psi | B | \psi \rangle| \leq 2^{-\text{dist}(A,B)/\xi} \]

\[ c(T) \leq T^{-\nu} e^{-\Delta/T}, \quad T \leq T_c \]
Area Law?

**Intuition:** Finite correlation length should imply area law

\[ l = O(\xi) \]

\[
\rho_{XZ} = \rho_X \otimes \rho_Z
\]

(Uhlmann)

\[
|\psi\rangle_{XYZ} = \left( U_{Y_1Y_2 \rightarrow Y} \otimes I_{XZ} \right) |\pi\rangle_{XY_1} |\nu\rangle_{Y_2Z}
\]
Area Law?

**Intuition:** Finite correlation length should imply area law

\[ l = O(\xi) \]

**Obstruction:** Data Hiding

\[ \rho_{x\psi} - s \rho_x \overline{\rho_x : Z} \rho_{xz} \leq 2^{-l/\xi}, \text{ but } \| \rho_{xz} - \rho_x \otimes \rho_z \|_1 \geq 1 \]
Area Law in 1D: A Success Story

- gap
  - (Has '04) \( \xi < O(1/\Delta) \)

- finite correlation length

- area law
  - (FNW '91 Vid '04)
    - (Hastings '07) \( S < e^{O(1/\Delta)} \)
    - (Arad et al '13) \( S < O(1/\Delta) \)

- MPS
Area Law in 1D: A Success Story

- **Gap**: $\xi < O(1/\Delta)$ (Has ’04)
- **Finite correlation length**: $S < e^{O(\xi)}$ (B, Hor ’13)
- **Area law**: $S < O(1/\Delta)$ (Arad et al ’13)
- **MPS**: $S < e^{O(1/\Delta)}$ (Hastings ’07)

(FNW ’91 Vid ’04)
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- **MPS**
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(Has ‘07) **Analytical** (Lieb-Robinson bound, filtering function, Fourier analysis)

(Arad et al ‘13) **Combinatorial** (Chebyshev polynomial)

(B., Hor ‘13) **Information-theoretical** (entanglement distillation, single-shot info theory)
Area Law in 1D: A Success Story

Efficient algorithm (Landau et al ‘14)

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Area Law in 1D: A Success Story

2nd guess: Area Law holds for

1. Groundstates of gapped Hamiltonians 1D, YES! >1D, OPEN
2. Any state with finite correlation length 1D, YES! >1D, OPEN
Area Law from Specific Heat

Statistical Mechanics 1.01

Gibbs state: \[ \rho_T := \frac{1}{Z_T} e^{-\frac{H}{T}}, \quad Z_T = \text{tr}(e^{-\frac{H}{T}}) \]
Area Law from Specific Heat

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**Gibbs state:** \[ \rho_T := \frac{1}{Z_T} e^{-H/T}, \quad Z_T = \text{tr}(e^{-H/T}) \]

**energy density:** \[ e(T) := \frac{1}{N} \text{tr}(H \rho_T) \]
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Area Law from Specific Heat

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\textbf{Specific heat capacity:} \[ c(T) := \left. \frac{\partial u(T')}{\partial T'} \right|_{T' = T} \]

\[ = \frac{1}{NT^2} \left( \text{tr}(H^2 \rho_T) - \text{tr}(H \rho_T)^2 \right) \]
Area Law from Specific Heat

Specific heat at $T$ close to zero:

Gapped systems: $c(T) \leq T^{-\nu} e^{-\Delta/T}$
(superconductor, Haldane phase, FQHE, ...)

Gapless systems: $c(T) \leq T^\gamma$
(conductor, ...)
**Thm** Let $H$ be a local Hamiltonian on a $d$-dimensional lattice $\Lambda := [n]^d$. Let $(R_1, ..., R_N)$, with $N = n^d/l^d$, be a partition of $\Lambda$ into cubic sub-lattices of size $l$ (and volume $l^d$).

1. Suppose $c(T) \leq T^{-\nu} e^{-\Delta/T}$ for every $T \leq T_c$. Then for every $\psi$ with $\langle \psi | H | \psi \rangle \leq n^d/l$

$$
\frac{1}{N} \sum_{i=1}^{N} S(\text{tr}_{\Lambda \setminus R_i}(|\psi\rangle\langle \psi|)) \leq O(l^{d-1} \log(l))
$$

$E_0(H) = 0$
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2. Suppose $c(T) \leq T^\nu$ for every $T \leq T_c$. Then for every $\psi$ with $\langle \psi | H | \psi \rangle \leq n^d/l$

\[
\frac{1}{N} \sum_{i=1}^{N} S(\text{tr}_{\Lambda \setminus R_i}(|\psi\rangle\langle\psi|)) \leq O(l^{d-1+\frac{1}{1+\nu}})
\]

$E_0(H) = 0$
Why?

Free energy:

\[
F_T(\sigma) := \text{tr}(H\sigma) - TS(\sigma)
\]

Variational Principle:

\[
F_T(\sigma) \geq F_T(\rho_T)
\]
Why?

Free energy:

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Let \( \pi := \text{tr}_{R_1}(|\psi\rangle\langle\psi|) \otimes \ldots \otimes \text{tr}_{R_1}(|\psi\rangle\langle\psi|) \)
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\text{tr}(\pi H) \leq \langle\psi|H|\psi\rangle + \left(\frac{n^d}{l^d}\right)c'l^{d-1} \leq cn^d/l
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By the variational principle, for \( T \) s.t. \( u(T) \leq c/l : S(\pi) \leq l^d S(T) \)
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\[ \text{tr}(\pi H) \leq \langle \psi | H | \psi \rangle + \left( \frac{n^d}{l^d} \right) c'l^{d-1} \leq cn^d/l \]

By the variational principle, for \( T \) s.t. \( u(T) \leq c/l : S(\pi) \leq l^d s(T) \)

Result follows from: \( s(T) = s(0) + \int_0^T \frac{c(T')}{{T'}} \, d{T'} \)
Summary and Open Questions

Summary:

Assuming the specific heat is “natural”, area law holds for every low-energy state of gapped systems and “subvolume law” for every low-energy state of general systems

Open questions:

- Can we prove a strict area law from the assumption on $c(T) \leq T^{-\nu} e^{-\Delta/T}$?
- Can we improve the subvolume law assuming $c(T) \leq T^\nu$?
- Are there natural systems violating one of the two conditions?
- Prove area law in $>1$D under assumption of (i) gap (ii) finite correlation length
- What else does an area law imply?