Entanglement Area Law (from Heat Capacity)

Fernando G.S.L. Brandão
University College London

Based on joint work arXiv:1410.XXXX with

Marcus Cramer
University of Ulm

Isfahan, September 2014

Plan

- What is an area law?
- Relevance
- Previous Work
- Area Law from Heat Capacity

Area Law

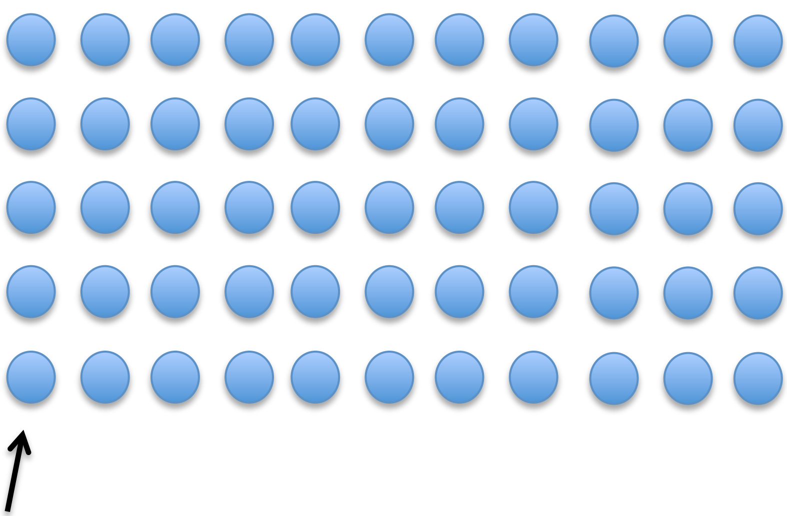
$|\psi\rangle \in \left(\mathbb{C}^2\right)^{\otimes n}$ 4^n parameters

Area Law

$$|\psi\rangle \in \left(\mathbb{C}^2\right)^{\otimes n}$$

4^n parameters

Quantum states on a lattice



$\mathbb{C}^2$

∂R : boundary of R

$|R|$: volume of R

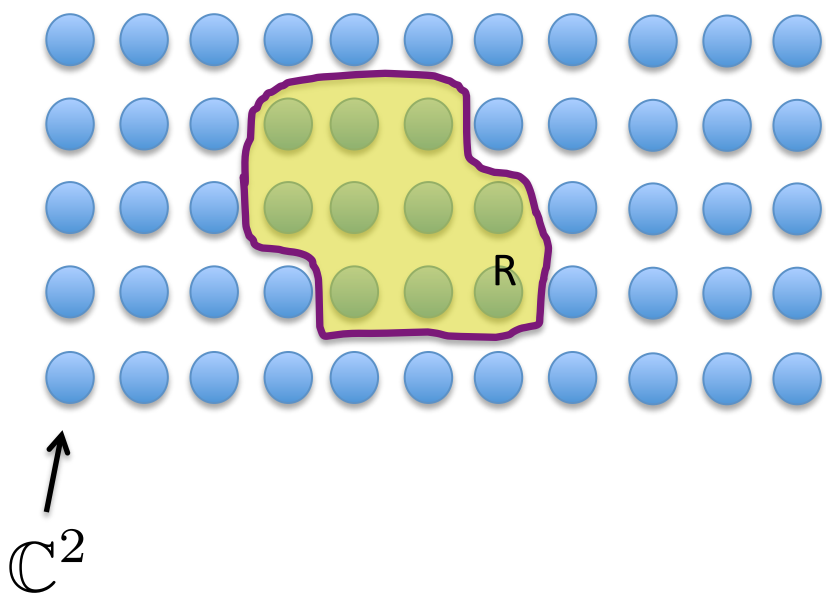
$|\partial R|$: volume of ∂R

Area Law

$$|\psi\rangle \in (\mathbb{C}^2)^{\otimes n}$$

4^n parameters

Quantum states on a lattice



∂R : boundary of R

$|R|$: volume of R

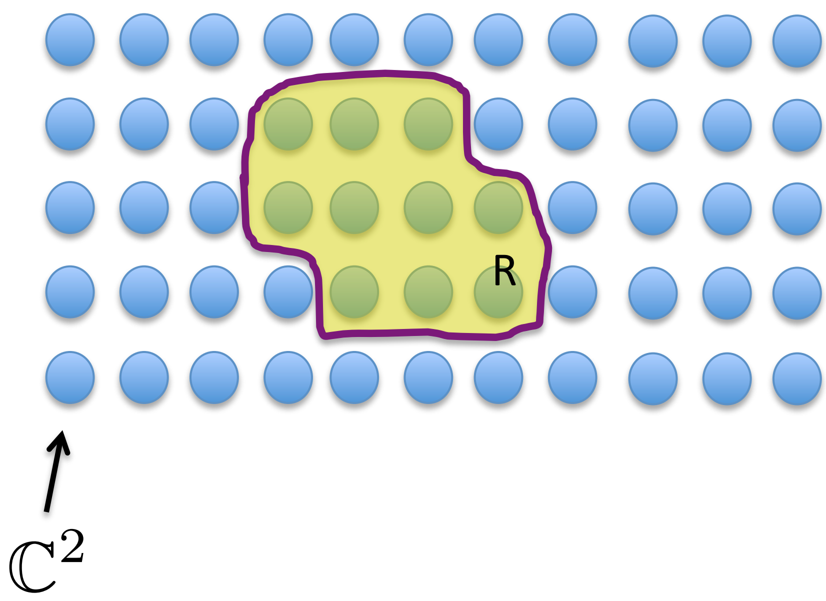
$|\partial R|$: volume of ∂R

Area Law

$$|\psi\rangle \in (\mathbb{C}^2)^{\otimes n}$$

4^n parameters

Quantum states on a lattice



Def: Area Law holds for $|\psi\rangle$ if for all R,

$$S(\mathrm{tr}_{R^c}(|\psi\rangle\langle\psi|)) \leq O(|\partial R|)$$

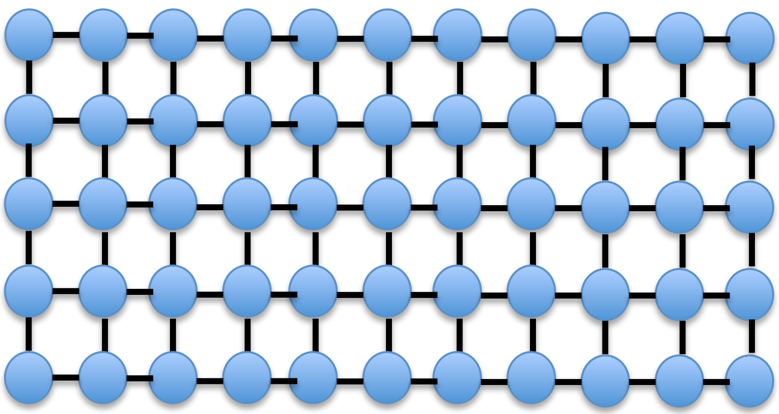
∂R : boundary of R

$|R|$: volume of R

$|\partial R|$: volume of ∂R

When does area law hold?

1st guess: it holds for every low-energy state of local models



$$H = \sum_{<i,j>} H_{i,j}, \qquad \|H_{i,j}\| \leq 1$$

$$\text{Energy of } |\psi\rangle \ : \ \langle\psi|H|\psi\rangle$$

$$H = \sum_k E_k |E_k\rangle\langle E_k|, \ \ E_0 \leq E_1 \leq \ldots$$

$$E_0 \ : \ \text{groundenergy}$$

$$|E_0\rangle \ : \ \text{groundstate}$$

When does area law hold?

1st ~~guess~~: it holds for every low-energy state of local models

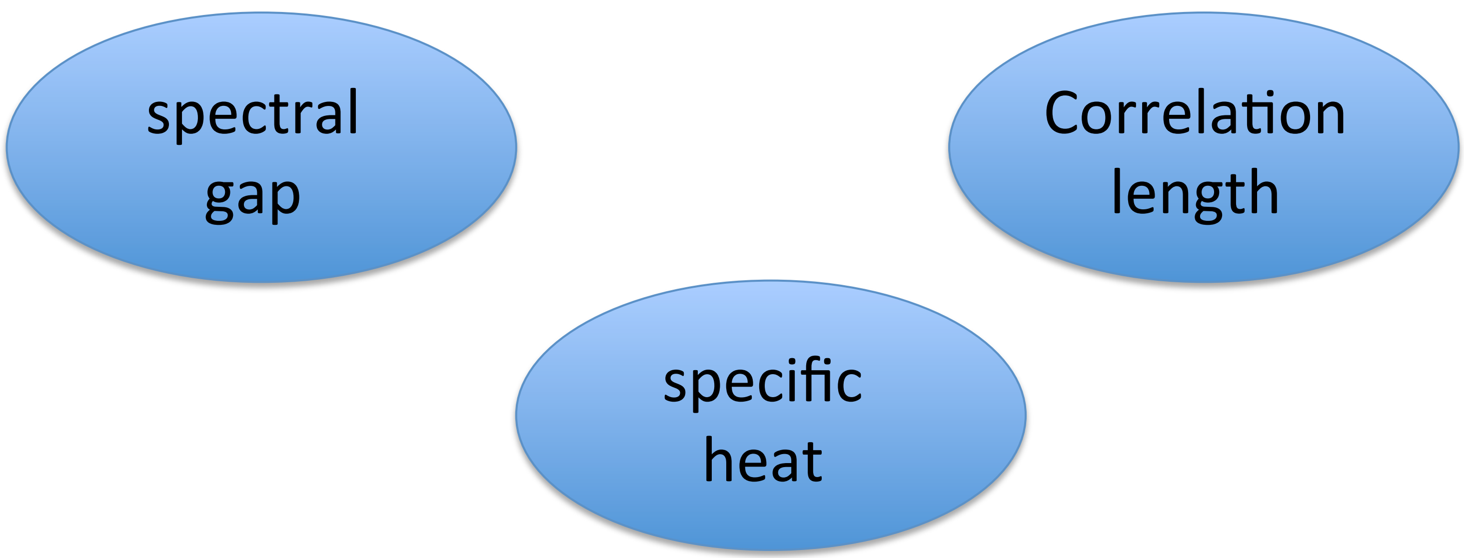
(Irani '07, Gotesman&Hastings '07)
There are 1D models with volume scaling of entanglement in groundstate

When does area law hold?

1st guess: it holds for every low-energy state of local models

(Irani '07, Gotesman&Hastings '07)
There are 1D models with volume scaling of entanglement in groundstate

Must put more restrictions on Hamiltonian/State!



Relevance

1D: Area Law S_α, $\alpha < 1$ → Good Classical Description (MPS)

(FNW '91
Vid '04)

Renyi Entropies: $$S_\alpha(\rho) := \frac{1}{1-\alpha} \log \mathrm{tr}(\rho^\alpha)$$

Matrix-Product-State: $$|\psi\rangle = \sum_{i_1,\ldots,i_n} \mathrm{tr}(A_{i_1} \ldots A_{i_n}) |i_1, \ldots, i_n\rangle$$

Relevance

1D:

Area Law
S_α, $\alpha < 1$

(FNW '91
Vid '04)

Good Classical
Description
(MPS)

Renyi Entropies:

$$S_\alpha(\rho) := \frac{1}{1-\alpha} \log \text{tr}(\rho^\alpha)$$

Matrix-Product-State:

$$|\psi\rangle = \sum_{i_1,\ldots,i_n} \text{tr}(A_{i_1} \ldots A_{i_n}) |i_1, \ldots, i_n\rangle$$

> 1D: ????

(appears to be connected with good tensor network description; e.g. PEPS, MERA)

Previous Work

(Bekenstein ‘73, Bombelli *et al* ‘86, ….)
Black hole entropy

(Vidal *et al* ‘03, Plenio *et al* ’05, …)
Integrable quasi-free bosonic systems and spin systems

⋮

see Rev. Mod. Phys. (Eisert, Cramer, Plenio ‘10)

Previous Work

(Bekenstein ‘73, Bombelli *et al* ‘86, ….)
Black hole entropy

(Vidal *et al* ‘03, Plenio *et al* ’05, …)
Integrable quasi-free bosonic systems and spin systems

⋮

see Rev. Mod. Phys. (Eisert, Cramer, Plenio ‘10)

2nd guess: Area Law holds for

1. Groundstates of gapped Hamiltonians
2. Any state with finite correlation length

Gapped Models

Def:

(gap) $\Delta(H) := E_1(H) - E_0(H)$

(gapped model) $\{H_n\}$ gapped if $\exists\ \Delta > 0,\ \Delta(H_n) \geq \Delta\ \forall\ n$

Gapped Models

Def:

(gap) $\Delta(H) := E_1(H) - E_0(H)$

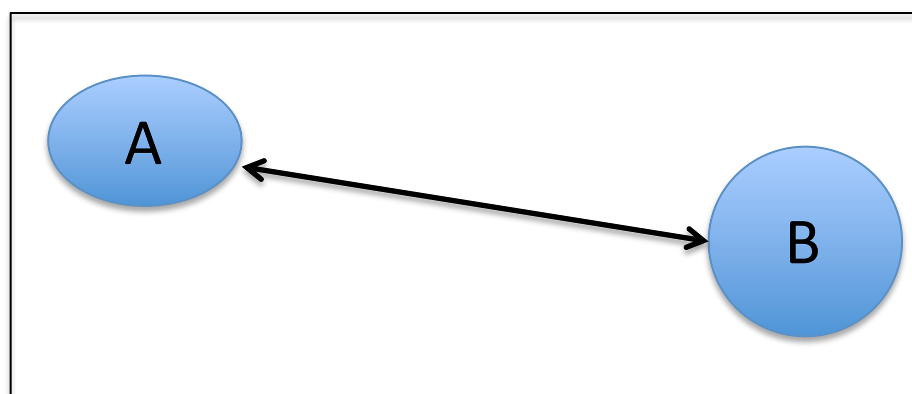
(gapped model) $\{H_n\}$ gapped if $\exists\, \Delta > 0,\ \Delta(H_n) \geq \Delta\ \forall\, n$

gap

(Has '04)

finite correlation length

$$|\langle\psi|A \otimes B|\psi\rangle - \langle\psi|A|\psi\rangle\langle\psi|B|\psi\rangle| \leq 2^{-\mathrm{dist}(A,B)/\xi}$$



Gapped Models

Def:

(gap) $\Delta(H) := E_1(H) - E_0(H)$

(gapped model) $\{H_n\}$ gapped if $\exists\, \Delta > 0,\ \Delta(H_n) \geq \Delta\ \forall\, n$

gap

(Has '04) → finite correlation length

$$|\langle\psi|A \otimes B|\psi\rangle - \langle\psi|A|\psi\rangle\langle\psi|B|\psi\rangle| \leq 2^{-\mathrm{dist}(A,B)/\xi}$$

(expectation no proof) → Exponential small heat capacity

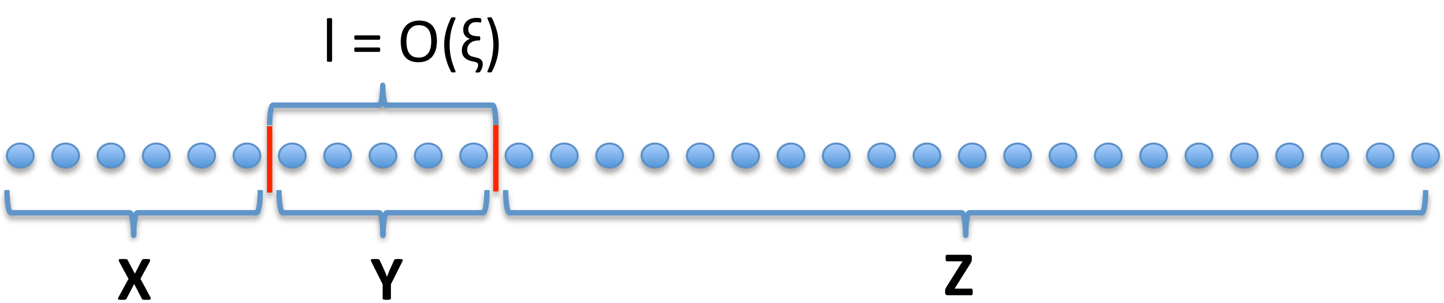
$$c(T) \leq T^{-\nu} e^{-\Delta/T}, \quad T \leq T_c$$

Area Law?

gap → finite correlation length
(Has '04)
ξ < O(1/Δ)

area law →(1D) MPS
(FNW '91
Vid '04)

Intuition: Finite correlation length should imply area law

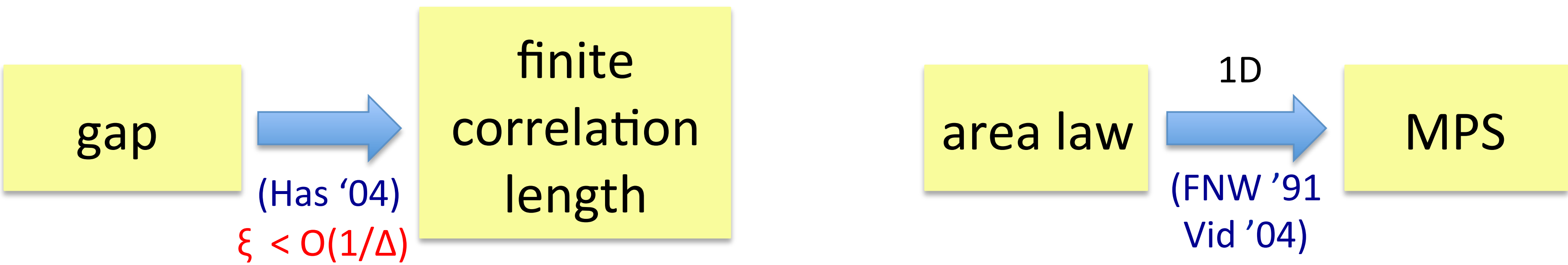


$$\rho_{XZ} = \rho_X \otimes \rho_Z$$

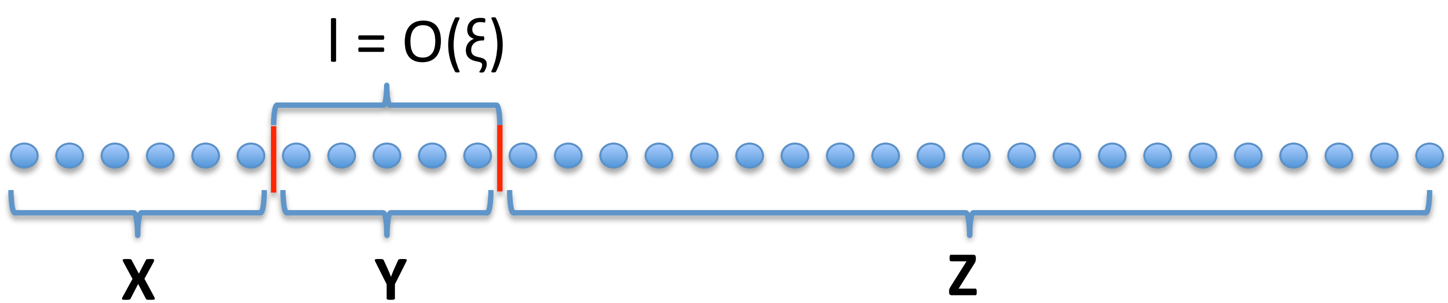
(Uhlmann) →

$$|\psi\rangle_{XYZ} = \left(U_{Y_1Y_2 \to Y} \otimes I_{XZ}\right)|\pi\rangle_{XY_1}|\upsilon\rangle_{Y_2Z}$$

Area Law?



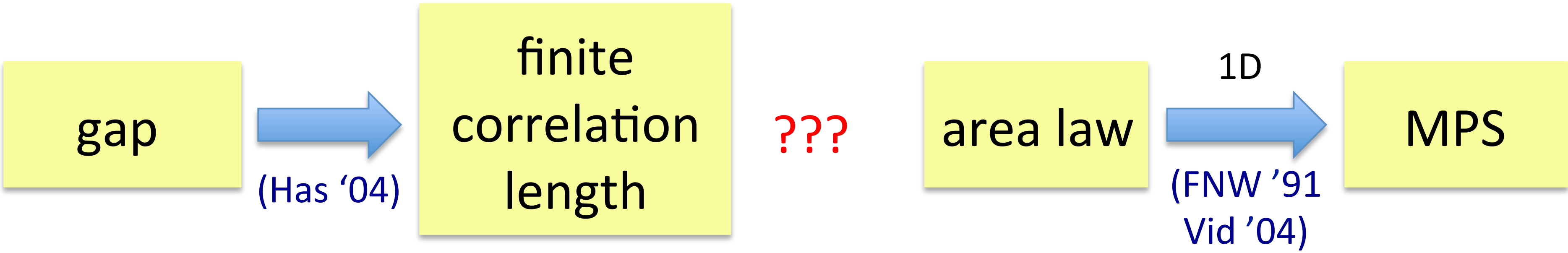
Intuition: Finite correlation length should imply area law



Obstruction: Data Hiding

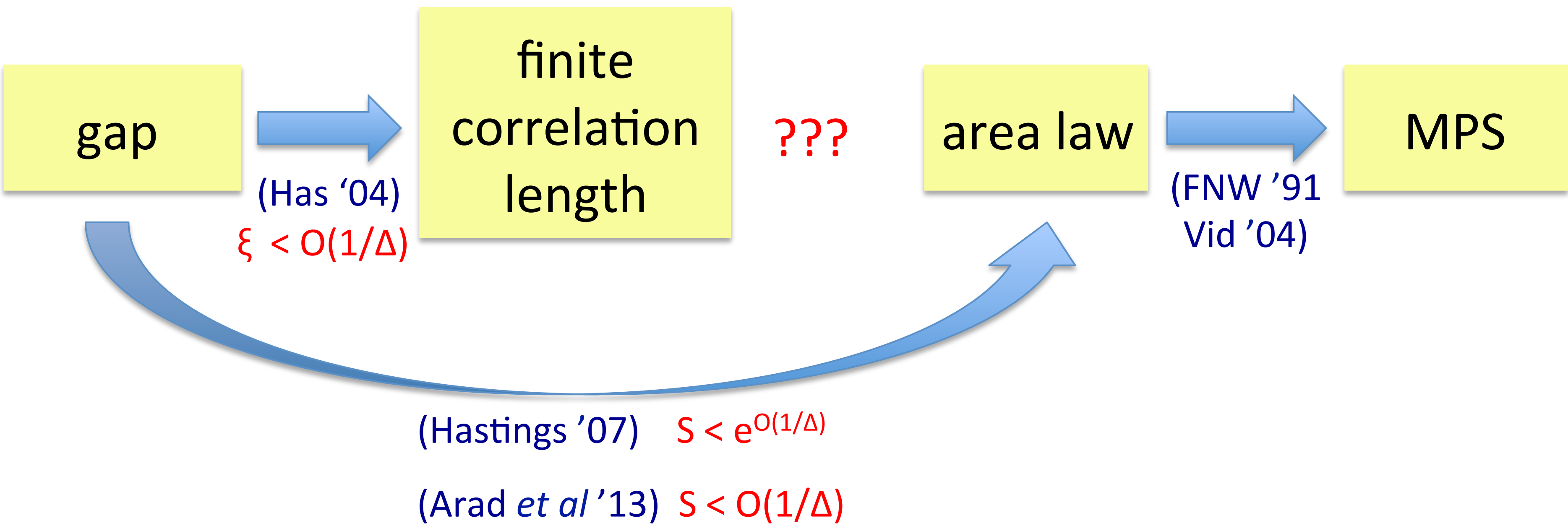
$\exists |\psi\rangle$ s.t. $\text{cor}(X : Z)_{\rho_{XZ}} \leq 2^{-l/\xi}$, but $\|\rho_{XZ} - \rho_X \otimes \rho_Z\|_1 \geq 1$

Area Law?

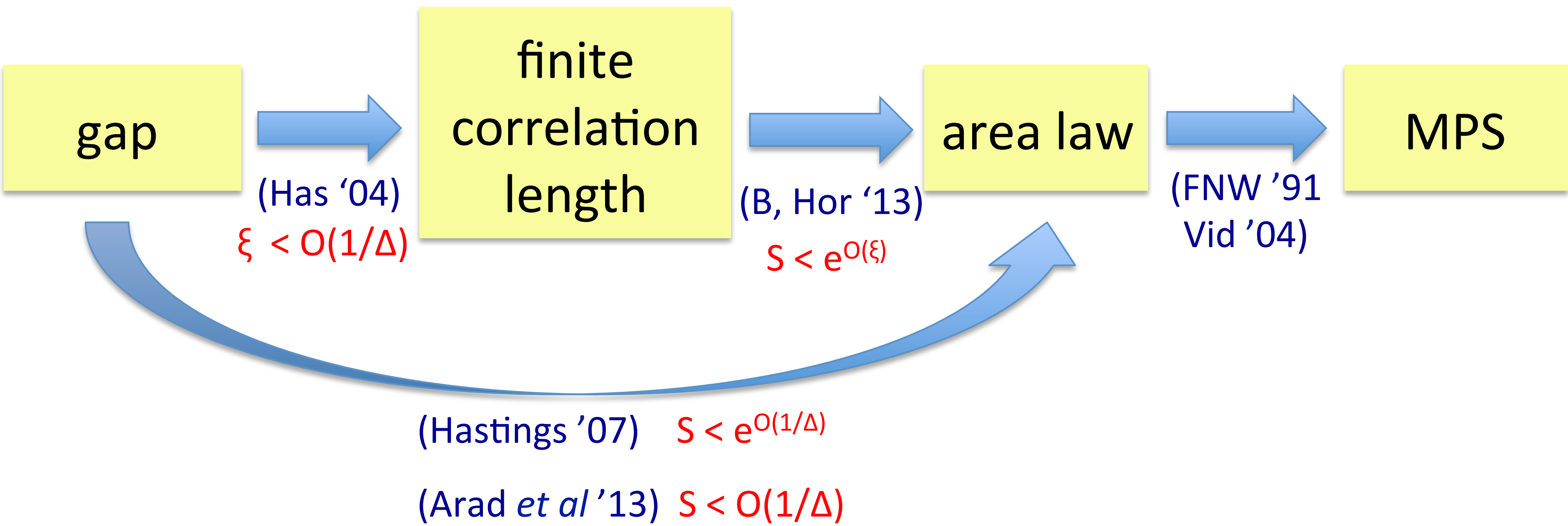


gap
(Has ‘04)
finite correlation length
???
area law
1D
(FNW ’91 Vid ’04)
MPS

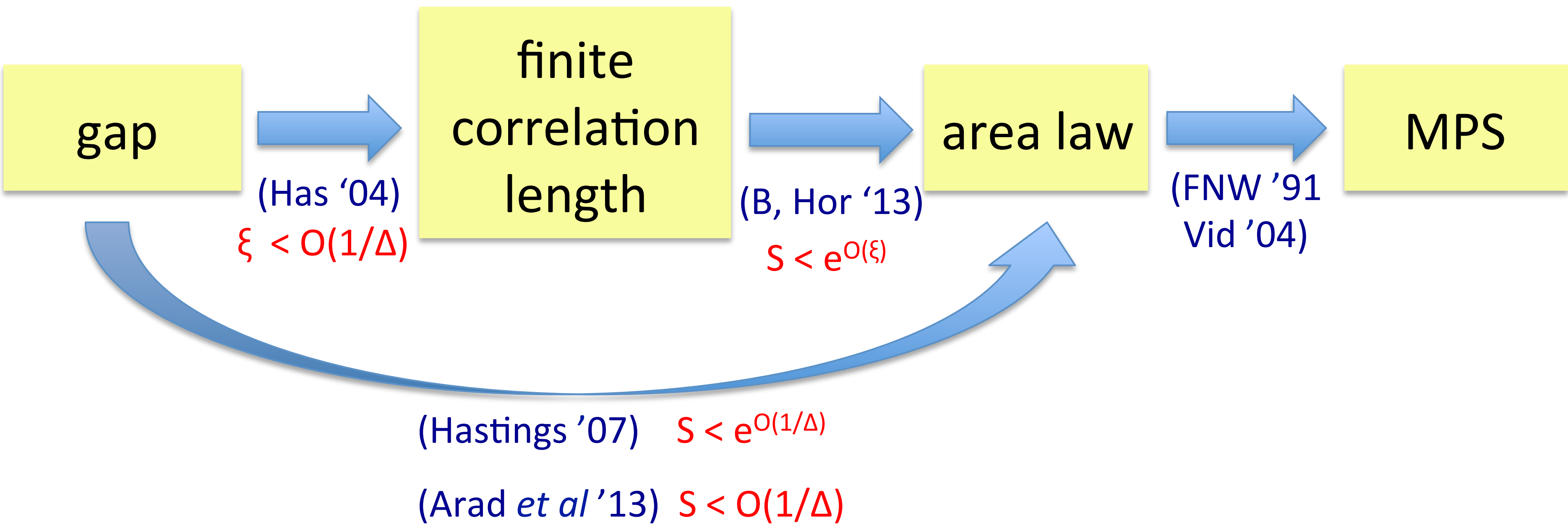
Area Law in 1D: A Success Story



Area Law in 1D: A Success Story



Area Law in 1D: A Success Story

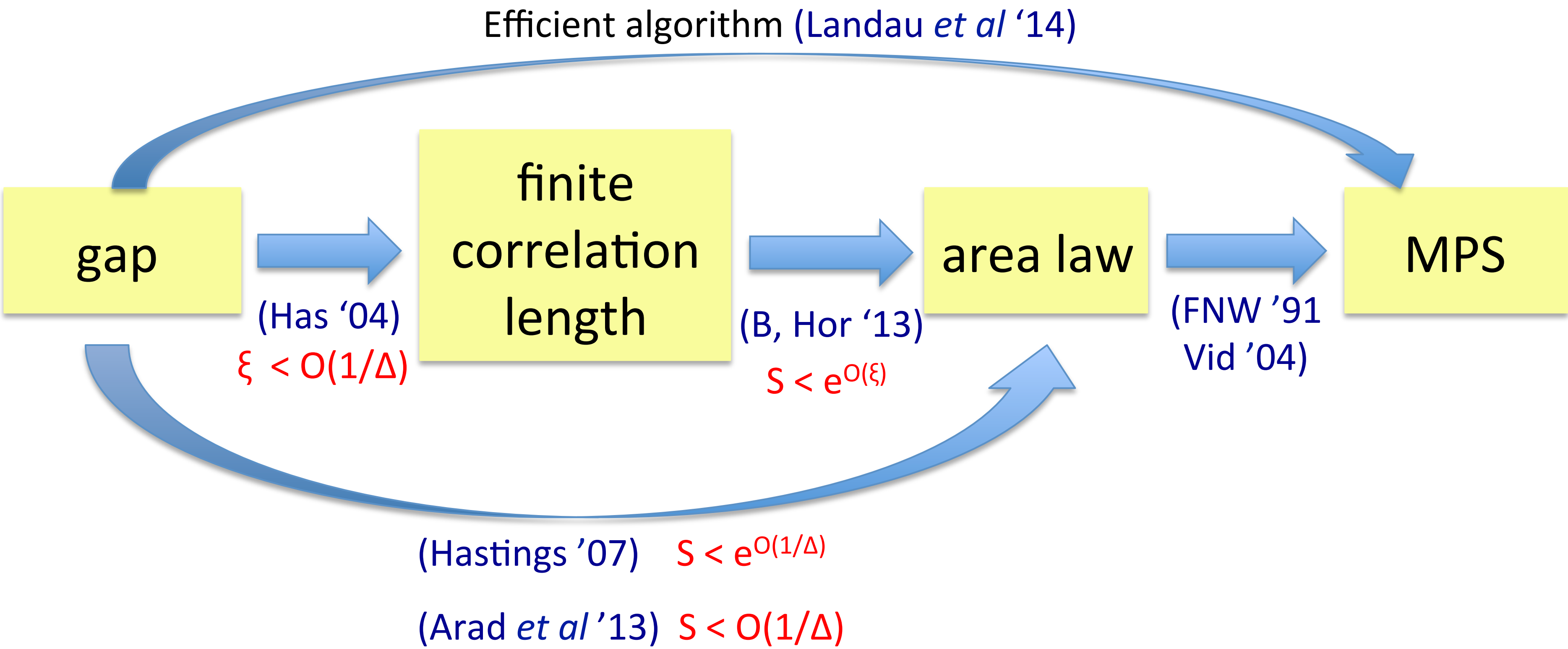


(Has '07) **Analytical** (Lieb-Robinson bound, filtering function, Fourier analysis)

(Arad *et al* '13) **Combinatorial** (Chebyshev polynomial)

(B., Hor '13) **Information-theoretical** (entanglement distillation, single-shot info theory)

Area Law in 1D: A Success Story

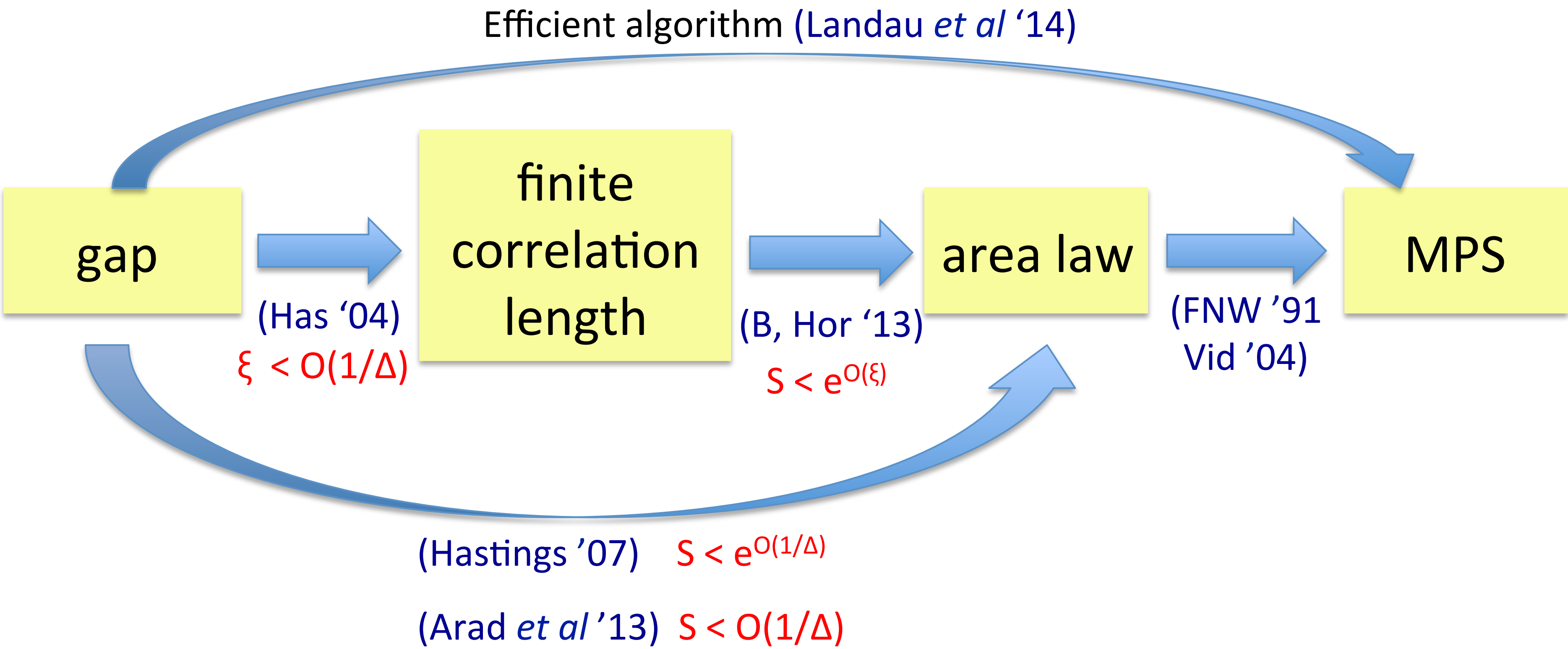


(Has ’07) **Analytical** (Lieb-Robinson bound, filtering function, Fourier analysis)

(Arad *et al* ’13) **Combinatorial** (Chebyshev polynomial)

(B., Hor ‘13) **Information-theoretical** (entanglement distillation, single-shot info theory)

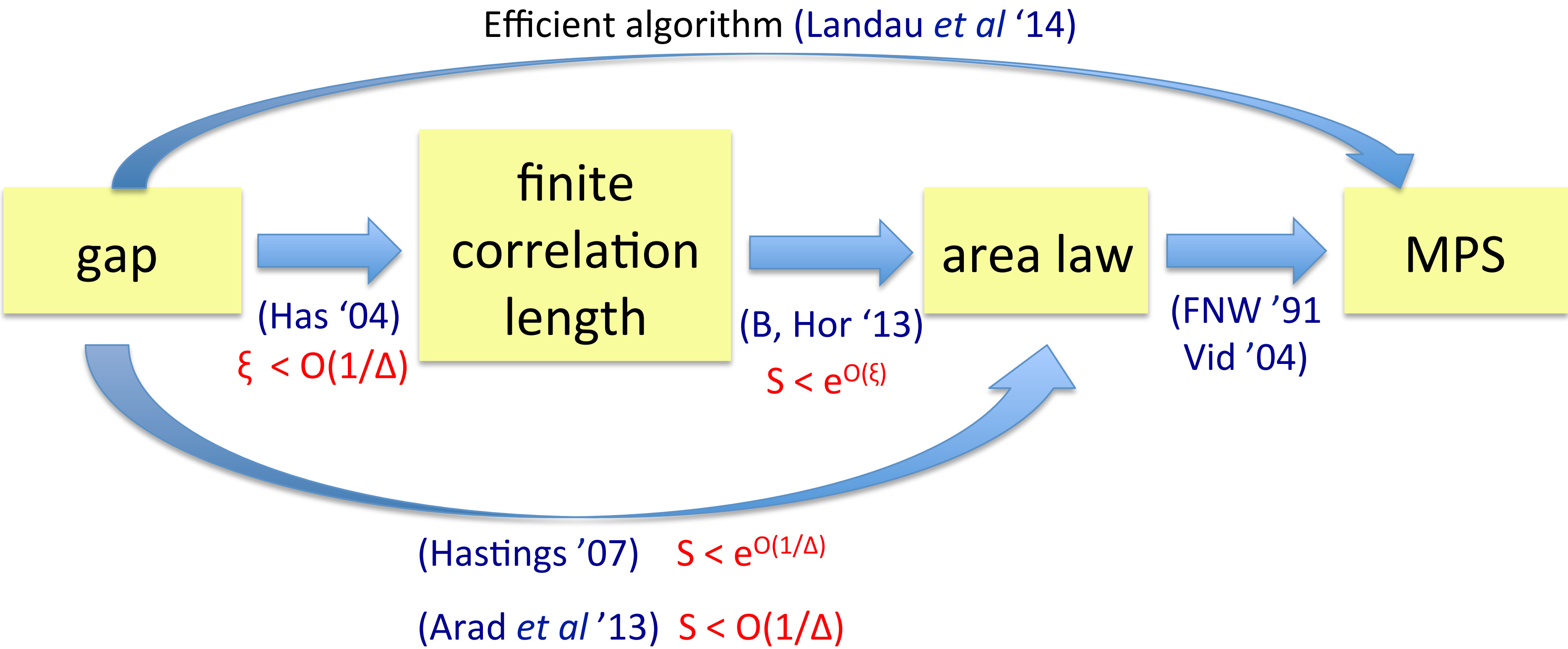
Area Law in 1D: A Success Story



2nd guess: Area Law holds for

1. Groundstates of gapped Hamiltonians
2. Any state with finite correlation length

Area Law in 1D: A Success Story



2nd guess: Area Law holds for

1. Groundstates of gapped Hamiltonians **1D, YES!** **>1D, OPEN**
2. Any state with finite correlation length **1D, YES!** **>1D, OPEN**

Area Law from Specific Heat

Statistical Mechanics 1.01

Gibbs state: $\rho_T := \frac{1}{Z_T} e^{-H/T}$, $Z_T = \mathrm{tr}(e^{-H/T})$

Area Law from Specific Heat

Statistical Mechanics 1.01

Gibbs state: $$\rho_T := \frac{1}{Z_T} e^{-H/T}, \quad Z_T = \mathrm{tr}(e^{-H/T})$$

energy density: $$e(T) := \frac{1}{N} \mathrm{tr}(H \rho_T)$$

Area Law from Specific Heat

Statistical Mechanics 1.01

Gibbs state: $\rho_T := \frac{1}{Z_T} e^{-H/T}, \;\; Z_T = \mathrm{tr}(e^{-H/T})$

energy density: $e(T) := \frac{1}{N} \mathrm{tr}(H \rho_T)$

entropy density: $s(T) := \frac{1}{N} S(\rho_T)$

Area Law from Specific Heat

Statistical Mechanics 1.01

Gibbs state: $\rho_T := \frac{1}{Z_T} e^{-H/T}, \; Z_T = \mathrm{tr}(e^{-H/T})$

energy density: $e(T) := \frac{1}{N} \mathrm{tr}(H \rho_T)$

entropy density: $s(T) := \frac{1}{N} S(\rho_T)$

Specific heat capacity:

$$c(T) := \left. \frac{\partial u(T')}{\partial T'} \right|_{T'=T} = \frac{1}{NT^2} \left(\mathrm{tr}(H^2 \rho_T) - \mathrm{tr}(H \rho_T)^2 \right)$$

Area Law from Specific Heat

Specific heat at *T* close to zero:

Gapped systems: $c(T) \leq T^{-\nu} e^{-\Delta/T}$

(superconductor,
Haldane phase,
FQHE, …)

Gapless systems: $c(T) \leq T^{\gamma}$

(conductor, …)

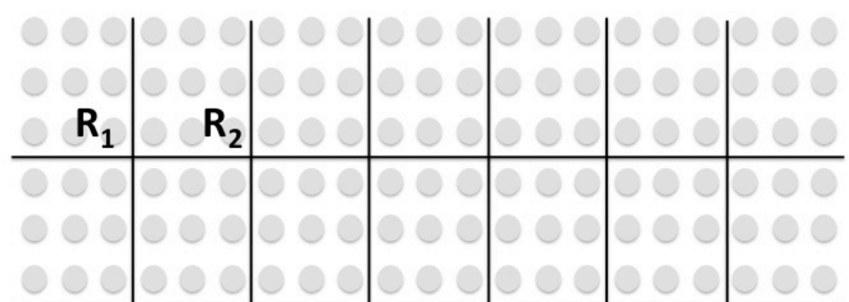
Area Law from Specific Heat

Thm Let *H* be a local Hamiltonian on a d-dimensional lattice $\Lambda := [n]^d$. Let $(R_1, \ldots, R_N)$, with $N = n^d/l^d$, be a partition of Λ into cubic sub-lattices of size *l* (and volume l^d).

1. Suppose $c(T) \leq T^{-\nu} e^{-\Delta/T}$ for every $T \leq T_c$. Then for every ψ with $\langle \psi | H | \psi \rangle \leq n^d / l$

$$\frac{1}{N} \sum_{i=1}^{N} S(\mathrm{tr}_{\Lambda \backslash R_i}(|\psi\rangle\langle\psi|)) \leq O(l^{d-1} \log(l))$$

$$E_0(H) = 0$$



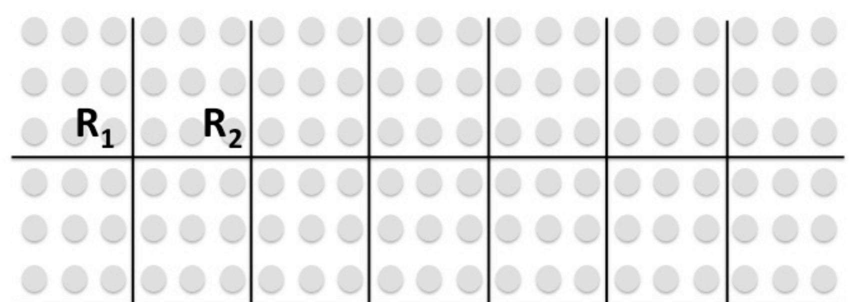
Area Law from Specific Heat

Thm Let *H* be a local Hamiltonian on a d-dimensional lattice $\Lambda := [n]^d$. Let $(R_1, \ldots, R_N)$, with $N = n^d/l^d$, be a partition of Λ into cubic sub-lattices of size *l* (and volume l^d).

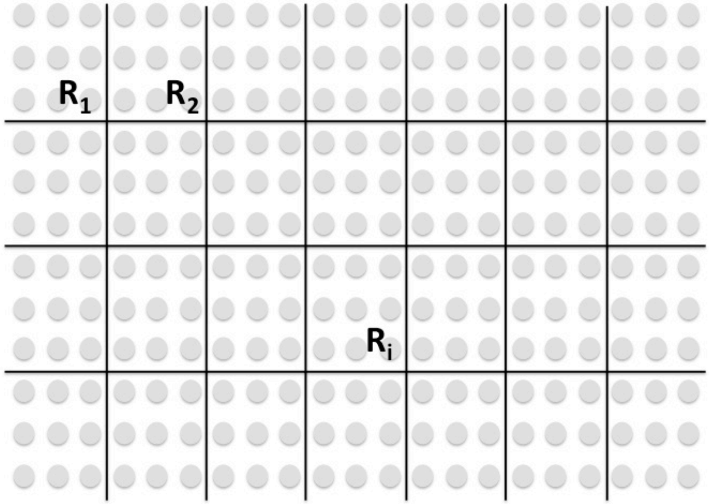
2. Suppose $c(T) \leq T^\nu$ for every $T \leq T_c$. Then for every ψ with
$\langle\psi|H|\psi\rangle \leq n^d/l$

$$\frac{1}{N}\sum_{i=1}^{N} S(\mathrm{tr}_{\Lambda\setminus R_i}(|\psi\rangle\langle\psi|)) \leq O(l^{d-1+\frac{1}{1+\nu}})$$

$$E_0(H) = 0$$



Why?



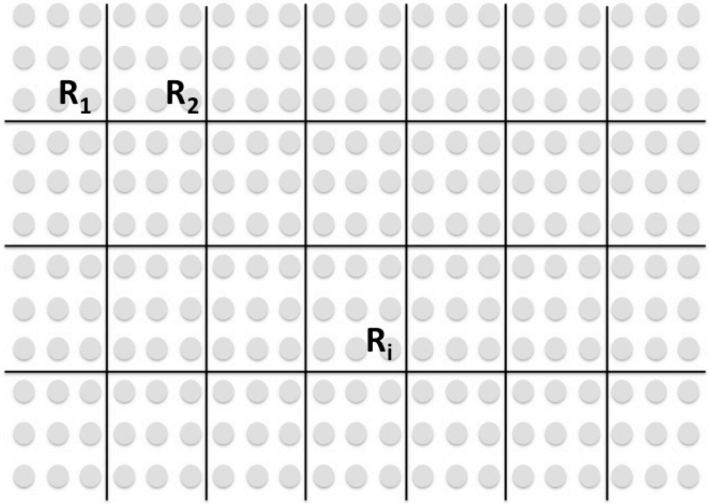
Free energy:

$$F_T(\sigma) := \mathrm{tr}(H\sigma) - TS(\sigma)$$

Variational Principle:

$$F_T(\sigma) \geq F_T(\rho_T)$$

Why?



Free energy:

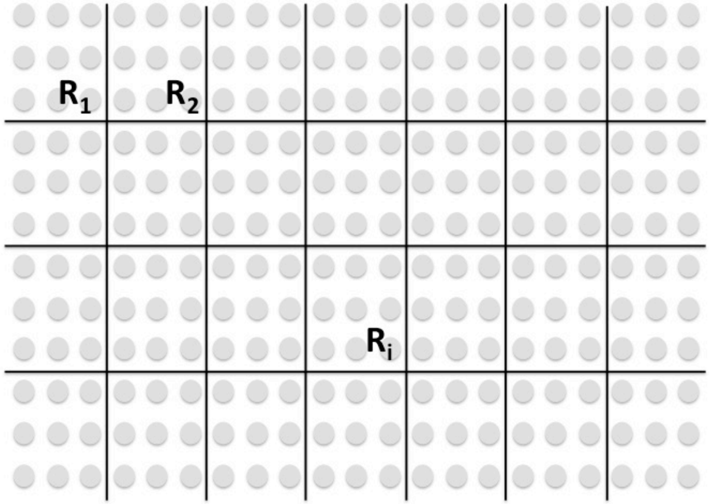
$$F_T(\sigma) := \mathrm{tr}(H\sigma) - TS(\sigma)$$

Variational Principle:

$$F_T(\sigma) \geq F_T(\rho_T)$$

Let $\pi := \mathrm{tr}_{\setminus R_1}(|\psi\rangle\langle\psi|) \otimes \ldots \otimes \mathrm{tr}_{\setminus R_1}(|\psi\rangle\langle\psi|)$

Why?



Free energy:

$$F_T(\sigma) := \mathrm{tr}(H\sigma) - TS(\sigma)$$

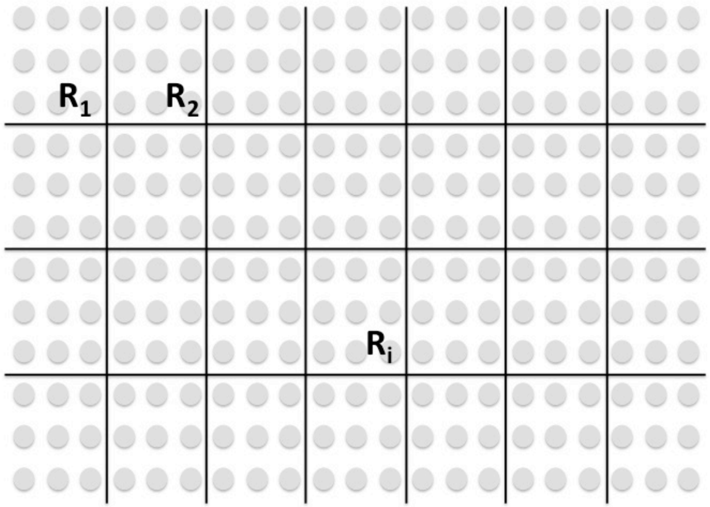
Variational Principle:

$$F_T(\sigma) \geq F_T(\rho_T)$$

Let $\pi := \mathrm{tr}_{\backslash R_1}(|\psi\rangle\langle\psi|) \otimes \ldots \otimes \mathrm{tr}_{\backslash R_1}(|\psi\rangle\langle\psi|)$

$$\mathrm{tr}(\pi H) \leq \langle\psi|H|\psi\rangle + \left(\frac{n^d}{l^d}\right) c' l^{d-1} \leq cn^d/l$$

Why?



Free energy:

$$F_T(\sigma) := \mathrm{tr}(H\sigma) - TS(\sigma)$$

Variational Principle:

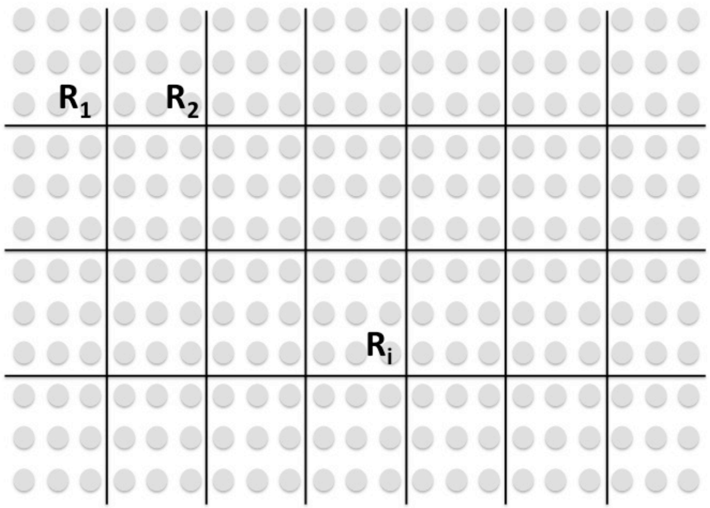
$$F_T(\sigma) \geq F_T(\rho_T)$$

Let $\pi := \mathrm{tr}_{\setminus R_1}(|\psi\rangle\langle\psi|) \otimes \ldots \otimes \mathrm{tr}_{\setminus R_1}(|\psi\rangle\langle\psi|)$

$$\mathrm{tr}(\pi H) \leq \langle\psi|H|\psi\rangle + \left(\frac{n^d}{l^d}\right) c' l^{d-1} \leq c n^d / l$$

By the variational principle, for *T* s.t. u(T) ≤ c/l : $S(\pi) \leq l^d s(T)$

Why?



Free energy:

$$F_T(\sigma) := \mathrm{tr}(H\sigma) - TS(\sigma)$$

Variational Principle:

$$F_T(\sigma) \geq F_T(\rho_T)$$

Let $\pi := \mathrm{tr}_{\backslash R_1}(|\psi\rangle\langle\psi|) \otimes \ldots \otimes \mathrm{tr}_{\backslash R_1}(|\psi\rangle\langle\psi|)$

$$\mathrm{tr}(\pi H) \leq \langle\psi|H|\psi\rangle + \left(\frac{n^d}{l^d}\right) c' l^{d-1} \leq c n^d / l$$

By the variational principle, for T s.t. u(T) ≤ c/l : $S(\pi) \leq l^d s(T)$

Result follows from: $s(T) = s(0) + \int_0^T \frac{c(T')}{T'} dT'$

Summary and Open Questions

Summary:

Assuming the specific heat is "natural", area law holds for every low-energy state of gapped systems and "subvolume law" for every low-energy state of general systems

Open questions:

- Can we prove a strict area law from the assumption on $c(T) \leq T^{-\nu} e^{-\Delta/T}$?
- Can we improve the subvolume law assuming $c(T) \leq T^{\nu}$?
- Are there natural systems violating one of the two conditions?
- Prove area law in >1D under assumption of (i) gap (ii) finite correlation length
- What else does an area law imply?