Physics 127C – Homework Set 4 (due June 15)

Problem 1

Let H be a lattice Hamiltonian on a finite dimensional lattice with eigen-decomposition $H = \sum_k E_k |E_k\rangle\langle E_k|$. A Hamiltonian H and an observable O satisfies the Eigenstate Thermalization Hypothesis if

$$\langle E_k|O|E_l\rangle = f_O(\overline{E})\delta_{kl} + 2^{-S(\overline{E})/2} g_O(\overline{E}, \Delta) r_{k,l}, \tag{1}$$

with $\overline{E} = (E_k + E_l)/2$, $\Delta = E_k - E_l$, $S(\overline{E})$ the thermodynamic entropy at energy $\overline{E}$, $r_{k,l}$ normal variables with mean 0 and variance 1, and f, g smooth functions. Define

$$O(t) = \langle \psi_0 | e^{-itH} O e^{itH} | \psi_0 \rangle \tag{2}$$

as the expectation value of O at time t on the initial state $|\psi_0\rangle$. Define the fluctuation of $O(t)$ as

$$\sigma_O^2 = \lim_{t_0 \to \infty} \frac{1}{t_0} \int_0^{t_0} dt [O(t)]^2 - (\overline{O})^2, \tag{3}$$

with

$$\overline{O} = \lim_{t_0 \to \infty} \frac{1}{t_0} e^{-itH} O e^{itH}. \tag{4}$$

the time-average of O.

Show ETH implies that

$$\sigma_O^2 \leq 2^{-S(\overline{E})} \max_{\overline{E}, \Delta} |g_O(\overline{E}, \Delta)|^2 \tag{5}$$

Since one can show that $g_O(\overline{E}, \Delta) \leq n^{1/2}$, with n the number of particles of the Hamiltonian, argue that Eq. (5) shows that fluctuations are exponentially small in n.

Problem 2

A phenomenological model of a many-body localized system of a one-dimensional model with local two-state degrees of freedom is the following:

$$H = \sum_i h_i \tau_i^z + \sum_{ij} J_{ij} \tau_i^z \tau_j^z + \ldots \tag{6}$$

where τ_i are local Pauli operators for the l-bit and denote the local integrals of motion within a localization length ξ. The coefficients h_i are random Zeeman fields uniformly distributed between $[-h, h]$. The couplings $J_{ij} = \tilde{J}_{ij} \exp(-|i-j|/\xi)$ describes the interaction between different l-bits and $\tilde{J}_{ij}$ are uniformly distributed between $[-J, J]$. The eigenstate of H can be written as $|n\rangle = |\tau_1^z, \tau_2^z, \ldots\rangle$, where $\tau_i^z = 0$ or 1.

Consider the infinite temperature out-of-time correlator given by

$$F(t) = \frac{1}{2^n} \sum_n \langle n | U^\dagger \tau_i^x U \tau_j^x U^\dagger \tau_i^x U \tau_j^x | n \rangle, \tag{7}$$

with $U = e^{-itH}$ and n the number of sites. Show (ignoring 3- and higher-order body terms in Eq. (6)) that

$$F(t) = \cos(4 J_{ij} t). \tag{8}$$

Also show that averaging over the disorder (i.e. the random couplings of the Hamiltonian),

$$\overline{F}(t) = \frac{\sin(4J\exp(-|i-j|/\xi)t)}{4J\exp(-|i-j|/\xi)t}. \tag{9}$$

Consider the early time behaviour of $\overline{F}(t)$ and show that it deviates from unit as a power law (in contrast to exponentially, as in ergodic systems).

Problem 3

Let H be a Hamiltonian with non-degenerate energy gaps. Let S be a region of the lattice and B its complement. Let $|\Psi(t)\rangle = e^{itH}|\Psi(0)\rangle$ be the state of the total system(i.e. SB) at time t and $\rho_S(t) = \mathrm{tr}_B(|\Psi(t)\rangle\langle\Psi(t)|)$ the reduced density matrix in S. Let also ω be the time averaged state. In class we saw that

$$\|\rho_S(t) - \omega_S\|_1 \leq \sqrt{\frac{d_S^2}{d^{\mathrm{eff}}(\omega)}}, \tag{10}$$

with

$$d^{\mathrm{eff}}(\omega) = \frac{1}{\mathrm{tr}(\omega^2)} \tag{11}$$

the effective dimension.

Show that the average of $d^{\mathrm{eff}}(\omega)$ for uniformly distributed initial states, over a subspace $\mathcal{H}_R$ of dimension d_R, satisfies:

$$\overline{d^{\mathrm{eff}}} \geq d_R/2, \tag{12}$$

Note: You can use without a proof the following fact (which is an easy exercise in representation theory):

$$\int \mu(d\psi)|\psi\rangle\langle\psi| \otimes |\psi\rangle\langle\psi| = (I+F)/(d(d+1)), \tag{13}$$

with μ the uniform (Haar) distribution over a vector space H of dimension d, I the identity on $\mathcal{H}\otimes\mathcal{H}$ and F the swap operator on the space $\mathcal{H}\otimes\mathcal{H}$, defined by

$$F|a,b\rangle = |b,a\rangle \tag{14}$$