Physics 127C – Homework Set 2 (due May 02)

Problem 1: Quantum Ising Model in a Transverse Field

Consider the quantum Ising model in a transverse field defined on the sites of a hypercubic d-dimensional lattice as in class:

$$H = -J \sum_{\langle i,j \rangle} \sigma_i^z \sigma_j^z - h \sum_i \sigma_i^x. \tag{1}$$

1. Consider first the paramagnetic phase with $J = 0$ where the ground state consists of all spins pointing in the x-direction. Let $|i\rangle$ denote a state with a single spin, at site i, flipped to point in the $-x$-direction. With $J = 0$ this is an eigenstate with energy independent of the location of the flipped spin. Define a projection operator into this degenerate manifold of states:

$$P = \sum_i |i\rangle\langle i|. \tag{2}$$

   Obtain an explicit expression for the full Hamiltonian when projected into this degenerate manifold, $H' = PHP$.

2. In order to split the degeneracy of the spin-flipped states to leading order in $J << h$ (using first order degenerate perturbation theory), requires diagonalizing the perturbation in the degenerate manifold. Using your projected Hamiltonian show that the plane wave state

$$|\mathbf{k}\rangle = \frac{1}{\sqrt{N}} \sum_i e^{-i\mathbf{k}.\mathbf{x}_i} |i\rangle, \tag{3}$$

   is in fact an eigenstate, $H'|\mathbf{k}\rangle = \varepsilon_k |\mathbf{k}\rangle$. Deduce the energy spectrum of the excited states, ε_k.

3. Next consider the ferromagnetic state with $h = 0$ where a ground state consists of all spins aligned along the (plus, say) z-direction. Using perturbation theory compute the shift in the ground state energy to second order in the transverse field h.

4. When $h = 0$ an exact degenerate manifold of excited eigenstates can be obtained by starting with the fully polarized Ferromagnetic state and flipping one spin at site i, which we will again denote as $|i\rangle$ (despite the degenerate notation, do not confuse this spin-flipped state in the FM with the spin-flipped states in the PM). To understand how this degeneracy will be split by small h, one can use second order degenerate perturbation theory. With the first order term vanishing, $\langle i|H_h|j\rangle = 0$, the second order shift can be obtained by diagonalizing the effective Hamiltonian:

$$H_{i,j}^{eff} = \sum_n \frac{\langle i|H_h|n\rangle\langle n|H_h|j\rangle}{\varepsilon_0 - E_n}, \tag{4}$$

   where ϵ_0 is the energy of the degenerate manifold (relative to the ground state) and the primed summation is over a complete set of unperturbed eigenstates (with energy E_n) excluding the states in the degenerate manifold. By computing the matrix elements and performing the summation, obtain an explicit expression for $H_{i,j}^{eff}$ in general dimension d.

5. The full effective Hamiltonian projected into the degenerate manifold is

$$H = \sum_i \langle i|H_J|i\rangle + \sum_{i,j} H_{ij}^{eff} |i\rangle\langle j|. \tag{5}$$

   Demonstrate explicitly that a place wave state is an exact eigenstate of H, and compute the corresponding energy as a function of momentum k. Noting that the shift in the ground state energy that you computed in (c) above, extract finally the excitation energy, ε_k of the spin-wave excitation in the Ferromagnetic state.

Problem 2: Quantum XY Model

Another quantum spin model which arises in various contexts is the so-called XY model, with Hamiltonian

$$H_{XY} = -J \sum_{\langle i,j \rangle} (\sigma_i^x \sigma_j^x + \sigma_i^y \sigma_j^y) - h \sum_i \sigma_i^z, \tag{6}$$

where the summation in the first term is over near-neighbor sites only.

1. Consider the unitary Operator

$$U = \prod_i e^{i\phi\sigma_i^z/2} = e^{-\phi\sigma_{tot}^z/2}, \tag{7}$$

   with $\sigma_{tot}^z = \sum_i \sigma_i^z$ the z-component of the total spin, which rotates all the spins by an angle ϕ around the z-axis. Show that the Hamiltonian H_{XY} commutes with U. This means it is possible to simultaneously diagonalize the Hamiltonian and the z-component of the total spin.

2. It is convenient to define spin raising and lowering operators, $\sigma_i^{\pm} = (\sigma_i^x + \pm i\sigma_i^y)/2$. Re-express the Hamiltonian H_{XY} in terms of $\sigma_i^{\pm}, \sigma_i^z$.

3. For the remainder of this problem we will specialize to a one-dimensional lattice. Consider the Jordan-Wigner transformation, a mapping in one-dimension between spin-1/2 operators and Fermions;

$$\sigma_i^+ = \prod_{j<i}(1 - 2c_j^\dagger c_j)c_i; \quad \sigma_i^- = \prod_{j<i}(1 - 2c_j^\dagger c_j)c_i^\dagger; \quad \sigma_i^z = (1 - 2c_i^\dagger c_i). \tag{8}$$

   Here, $c_i^\dagger, c_i$ are Fermions, satisfying the canonical anti-commutation relations. Using the Jordan-Wigner transformation re-write the Hamiltonian H_{XY} in terms of the Fermion operators.

4. It is convenient to introduce the Fourier transform of the Fermion operators as follows:

$$\hat{c}_k = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} c_i e^{-kx_i}, \tag{9}$$

   where $x_i = i = 1, 2, \ldots, N$ runs over N sites (assumed even) of the 1D lattice with periodic boundary conditions, and the momentum $k = 2\pi n/N$ with integer $n = -N/2 + 1,_n /2 + 2, \ldots, N/2$. Here $\hat{c}_k^\dagger$ is the creation operator for a Fermion in a momentum eigenstate k. Show that $\hat{c}_k$ satisfy the anti-commutation relations.

5. Show that the Hamiltonian, when re-expressed in terms of $\hat{c}_k, \hat{c}_k^\dagger$ can be put into a diagonal form:

$$H_{XY} = \sum_k E_k \hat{c}_k^\dagger \hat{c}_k, \tag{10}$$

   with $|k| \leq \pi$. Obtain an expression for the particle dispersion E_k.

6. The ground state of this free Fermion Hamiltonian corresponds to filling up all momentum states with negative energy, $E_k < 0$,

$$|G\rangle = \prod_{k:E_k<0} \hat{c}_k^\dagger |vac\rangle, \tag{11}$$

   where the prime denotes the restricted product over negative energy states only, and $|vac\rangle$ is the vacuum of Fermions that satisfies $\hat{c}_k|vac\rangle = 0$.

The magnetization of the spins along the z-axis is given by $M_z = (1/N)\sum_i \langle G|\sigma_i^z|G\rangle$. By re-expressing M_z in terms of Fermions using the Jordan-Wigner, compute the magnetizaton as a function of h/J. What is the critical value of the field, h_c, above which the magnetization is fully saturated at plus one (or minus one if h is negative)?

7. The magnetic susceptibility is defined as $\xi = \partial M_z/\partial h$. Obtain an expression for the susceptibility as a function of h, and show that the susceptibility diverges upon approaching the critical field from below as $\xi(h) \sim (h_c - h)^{-\gamma}$. What is the value of the critical exponent γ?