Physics 125b Problem Set 4 Solutions

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Problem 1

We use Fermi's Golden Rule,

$$R_{i\to f} = \frac{2\pi}{\hbar}|\langle f^0|H^1|i^0\rangle|^2\delta\left(E_f^0 - E_i^0 - \hbar\omega\right). \tag{1}$$

The interaction Hamiltonian is as in Shankar,

$$\begin{aligned} H^1(t) &= \frac{e}{2mc}e^{i\vec{k}\cdot\vec{r}}\left(\vec{A}_0\cdot\vec{P}\right)e^{-i\omega t} &(2)\\ &\equiv H^1 e^{-i\omega t}. &(3)\end{aligned}$$

The harmonic oscillator ground state is

$$\psi_i(r) = \left(\frac{m\omega}{\pi\hbar}\right)^{3/4} e^{-\frac{m\omega r^2}{2\hbar}}, \tag{4}$$

and the final state is a plane wave,

$$\psi_f(r) = \frac{1}{(2\pi\hbar)^{3/2}}e^{i\vec{p}_f\cdot\vec{r}/\hbar} \tag{5}$$

so that (ignoring the $e^{i\vec{k}\cdot\vec{r}}$ term) the matrix element is

$$|\langle f^0|H^1|i^0\rangle| = \frac{e}{2mc}\left(\frac{m\omega}{\pi\hbar}\right)^{3/4}\frac{1}{(2\pi\hbar)^{3/2}}\int e^{-i\vec{p}_f\cdot\vec{r}/\hbar}\vec{A}_0\cdot\left(-i\hbar\vec{\nabla}\right)e^{-\frac{m\omega r^2}{2\hbar}}r^2\sin\theta dr d\theta d\phi. \tag{6}$$

Integrating by parts gives

$$|\langle f^0|H^1|i^0\rangle| = \frac{e}{2mc}\left(\frac{m\omega}{\pi\hbar}\right)^{3/4}\frac{1}{(2\pi\hbar)^{3/2}}\vec{p}_f\cdot\vec{A}_0\int e^{-i\vec{p}_f\cdot\vec{r}/\hbar}e^{-\frac{m\omega r^2}{2\hbar}}r^2\sin\theta dr d\theta d\phi. \tag{7}$$

We compute the integral,

$$\int e^{-ip_f r\cos\theta/\hbar} e^{-\frac{m\omega r^2}{2\hbar}} r^2 \sin\theta dr d\theta d\phi = \left(\frac{2\pi\hbar}{m\omega}\right)^{3/2} e^{-\frac{p_f^2}{2m\omega\hbar}} \tag{8}$$

so that

$$|\langle f^0|H^1|i^0\rangle| = \frac{e}{2mc}\left(\frac{1}{\pi m\omega\hbar}\right)^{3/4} \vec{p}_f \cdot \vec{A}_0 \; e^{-\frac{p_f^2}{2m\omega\hbar}}. \tag{9}$$

Plugging in the rate we obtain

$$R_{i\to f} = \frac{2\pi}{\hbar}\left(\frac{e}{2mc}\right)^2 \frac{1}{(\pi m\omega\hbar)^{3/2}} \left(\vec{p}_f \cdot \vec{A}_0\right)^2 e^{-\frac{p_f^2}{m\omega\hbar}} \; \delta\left(E_f^0 - E_i^0 - \hbar\omega\right). \tag{10}$$

The δ-function fixes the momentum to be

$$p_f = \sqrt{2m\left(E_i^0 + \hbar\omega\right)} \tag{11}$$

and for simplicity we take E_i^0 to be the harmonic oscillator ground state shifted by the constant piece,

$$E_i^0 = \frac{3\hbar\omega}{2} - \frac{m\omega^2 r_0^2}{2}, \tag{12}$$

so that

$$p_f = \sqrt{m\left(5\hbar\omega - m\omega^2 r_0^2\right)} \tag{13}$$

and as in Shankar the rate becomes

$$R_{i\to d\Omega} = \frac{2\pi}{\hbar}\left(\frac{e}{2mc}\right)^2 \frac{mp_f}{(\pi m\omega\hbar)^{3/2}} \left(\vec{p}_f \cdot \vec{A}_0\right)^2 e^{-\frac{p_f^2}{m\omega\hbar}} d\Omega. \tag{14}$$

Integrating over solid angles gives

$$R_{i\to \mathrm{all}} = \frac{8\pi^2}{3\hbar}\left(\frac{e}{2mc}\right)^2 \frac{mp_f^3 A_0^2}{(\pi m\omega\hbar)^{3/2}} e^{-\frac{p_f^2}{m\omega\hbar}}. \tag{15}$$

Problem 2

From Eq. (19.3.8) in Shankar

$$\frac{d\sigma}{d\Omega} = \left|\frac{2mV_0}{\hbar^2}\int \frac{\sin qr}{q}\theta(r_0 - r)rdr\right|^2 \tag{16}$$

$$= \frac{4m^2V_0^2}{\hbar^4}\frac{\left(\sin qr_0 - qr_0\cos qr_0\right)^2}{q^6}. \tag{17}$$

The total cross-section is

$$\sigma = 2\pi \int_{\theta=0}^{\pi} \frac{d\sigma}{d\Omega} d\left(\cos\theta\right). \tag{18}$$

The trick is to write it in terms of q; since

$$q^2 = 2k^2\left(1 - \cos\theta\right), \tag{19}$$

we have

$$d\left(\cos\theta\right) = -\frac{qdq}{k^2} \tag{20}$$

so (up to a sign) the integral becomes

$$\sigma = \frac{8\pi m^2V_0^2}{\hbar^4k^2}\int_0^{2k}\frac{\left(\sin qr_0 - qr_0\cos qr_0\right)^2}{q^5}dq \tag{21}$$

$$= \frac{\pi m^2V_0^2}{\hbar^4}\frac{32k^4r_0^4 - 8k^2r_0^2 + 4kr_0\sin(4kr_0) + \cos(4kr_0) - 1}{16k^6}. \tag{22}$$

Sending $kr_0 \to 0$ is the same as sending $qr_0 \to 0$. Using that

$$\lim_{x\to 0}\frac{\left(\sin x - x\cos x\right)^2}{x^6} = \frac{1}{9} \tag{23}$$

we obtain

$$\frac{d\sigma}{d\Omega} = \frac{4m^2V_0^2r^6}{9\hbar^4}. \tag{24}$$

Since the θ dependence drops out (the scattering is isotropic) the total cross-section is obtained by multiplying with 4π.

Problem 3

Starting from the Born approximation for elastic scattering off a potential $V(\vec{x})$,

$$f(\vec{k}, \vec{k}') = -\frac{1}{4\pi}\frac{2m}{\hbar^2}\int d^3\vec{x}' e^{i(\vec{k}-\vec{k}')\cdot\vec{x}'}\ V(\vec{x}')\,. \tag{25}$$

Introducing the momentum transfer $\vec{q} = (\vec{k} - \vec{k}')$, with magnitude $q = |\vec{k} - \vec{k}'| = 2k \sin\frac{\theta}{2}$, as well as switching to spherical coordinates for the $d^3\vec{x}'$ integration (the scattering potential is spherically symmetric $V(r) = V_0 e^{-(r/r_0)^2}$), we find the following expression (choosing the z-direction of our $d^3\vec{x}'$ integration along $\vec{q}$):

$$f(\theta, \phi) = -\frac{1}{4\pi}\frac{2m}{\hbar^2}\int_0^\infty r'^2 dr' \int_{-1}^{+1} d\cos\theta' \int_0^{2\pi} d\phi' e^{iqr'\cos\theta'} V_0 e^{-(r'/r_0)^2}\,. \tag{26}$$

Integrating out the angular variables, we get,

$$f(\theta) = -\frac{2m}{\hbar^2}\frac{V_0}{q}\int_0^\infty dr' r' \sin(qr') e^{-(r'/r_0)^2}\,. \tag{27}$$

This integral can be done by writing the sin in terms of exponentials and completing the square (or with Mathematica directly),

$$f(\theta) = -\frac{\sqrt{\pi} m V_0 r_0^3}{2\hbar^2} e^{-\frac{(qr_0)^2}{4}}\,, \tag{28}$$

from which we obtain the differential cross section,

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2 = \frac{\pi m^2 V_0^2 r_0^6}{4\hbar^4} e^{-\frac{(qr_0)^2}{2}} \tag{29}$$

In order to get the total cross section σ, we integrate over all angles (θ, ϕ), taking into account that $q^2 = 2k^2(1 - \cos\theta)$,

$$\begin{aligned}\sigma &= \frac{\pi m^2 V_0^2 r_0^6}{4\hbar^4}\int_{-1}^{+1} d\cos\theta \int_0^{2\pi} d\phi\ e^{-k^2 r_0^2 (1-\cos\theta)} \\ &= \frac{\pi^2 m^2 V_0^2 r_0^6}{2\hbar^4}\frac{1 - e^{-2(kr_0)^2}}{(kr_0)^2}\end{aligned} \tag{30}$$