Physics 125b Solutions of Problem Set 3

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Problem 1

The perturbation breaks rotational invariance of the Hamiltonian. One can see this by e.g. evaluating the commutator $[J_x, H^{(1)}]$ and verifying that $[J_x, H^{(1)}] \neq 0$.

The eigenvectors of the unperturbed Hamiltonian will be indexed by the total energy of two 3D harmonic oscillators, E_{tot}. Note that 3D harmonic oscillator can be treated as three 1D oscillators, and thus the energy levels of fermion i can be labeled by a triple of numbers (n_i^x, n_i^y, n_i^z). Note that

$$z_i = \sqrt{\frac{\hbar}{2m\omega}}(a_i^z + (a_i^z)^\dagger) \tag{1}$$

When we evaluate first order corrections $\langle\psi|H^{(1)}|\psi\rangle$ to energy for the state $|\psi\rangle$ with total energy E_{tot}, the perturbation $H^{(1)}$ acting on $|\psi\rangle$ will create a superposition of states with different total energy, $E_{tot} \pm \hbar\omega$ (if $|\psi\rangle$ is the ground state, then only $E_{tot} + \hbar\omega$). We know that states with different energies are orthogonal and thus first order corrections vanish.

Problem 2

Note that the wavefunction describing the system of two identical fermions has to be antisymmetric. There is a unique ground state $|\psi_0\rangle$ with total energy $E_0 = 3\hbar\omega$, namely

$$|\psi_0\rangle = \frac{1}{\sqrt{2}}|0,0,0\rangle \otimes |0,0,0\rangle \otimes (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle). \tag{2}$$

There are 12 excited states $|\psi_i\rangle$ with total energy $E_i = 4\hbar\omega$ listed below

$$
\begin{aligned}
|\psi_1\rangle &= \frac{1}{2}(|1,0,0\rangle \otimes |0,0,0\rangle + |0,0,0\rangle \otimes |1,0,0\rangle) \otimes (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle),\\
|\psi_2\rangle &= \frac{1}{2}(|0,1,0\rangle \otimes |0,0,0\rangle + |0,0,0\rangle \otimes |0,1,0\rangle) \otimes (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle),\\
|\psi_3\rangle &= \frac{1}{2}(|0,0,1\rangle \otimes |0,0,0\rangle + |0,0,0\rangle \otimes |0,0,1\rangle) \otimes (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle),\\
|\psi_4\rangle &= \frac{1}{\sqrt{2}}(|1,0,0\rangle \otimes |0,0,0\rangle - |0,0,0\rangle \otimes |1,0,0\rangle) \otimes |\uparrow\uparrow\rangle,\\
|\psi_5\rangle &= \frac{1}{\sqrt{2}}(|0,1,0\rangle \otimes |0,0,0\rangle - |0,0,0\rangle \otimes |0,1,0\rangle) \otimes |\uparrow\uparrow\rangle,\\
|\psi_6\rangle &= \frac{1}{\sqrt{2}}(|0,0,1\rangle \otimes |0,0,0\rangle - |0,0,0\rangle \otimes |0,0,1\rangle) \otimes |\uparrow\uparrow\rangle,\\
|\psi_7\rangle &= \frac{1}{2}(|1,0,0\rangle \otimes |0,0,0\rangle - |0,0,0\rangle \otimes |1,0,0\rangle) \otimes (|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle),\\
|\psi_8\rangle &= \frac{1}{2}(|0,1,0\rangle \otimes |0,0,0\rangle - |0,0,0\rangle \otimes |0,1,0\rangle) \otimes (|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle),\\
|\psi_9\rangle &= \frac{1}{2}(|0,0,1\rangle \otimes |0,0,0\rangle - |0,0,0\rangle \otimes |0,0,1\rangle) \otimes (|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle),\\
|\psi_{10}\rangle &= \frac{1}{\sqrt{2}}(|1,0,0\rangle \otimes |0,0,0\rangle - |0,0,0\rangle \otimes |1,0,0\rangle) \otimes |\downarrow\downarrow\rangle,\\
|\psi_{11}\rangle &= \frac{1}{\sqrt{2}}(|0,1,0\rangle \otimes |0,0,0\rangle - |0,0,0\rangle \otimes |0,1,0\rangle) \otimes |\downarrow\downarrow\rangle,\\
|\psi_{12}\rangle &= \frac{1}{\sqrt{2}}(|0,0,1\rangle \otimes |0,0,0\rangle - |0,0,0\rangle \otimes |0,0,1\rangle) \otimes |\downarrow\downarrow\rangle.
\end{aligned} \tag{3}
$$

Note that

$$
\begin{aligned}
H^{(1)}|\psi_0\rangle =& \frac{\epsilon\hbar}{2}\sqrt{\frac{\hbar}{4m\omega}}|0,0,1\rangle \otimes |0,0,0\rangle \otimes (-|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) + \\
& \frac{\epsilon\hbar}{2}\sqrt{\frac{\hbar}{4m\omega}}|0,0,0\rangle \otimes |0,0,1\rangle \otimes (|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \\
=& \frac{\epsilon\hbar}{2}\sqrt{\frac{\hbar}{4m\omega}}(-|0,0,1\rangle \otimes |0,0,0\rangle + |0,0,0\rangle \otimes |0,0,1\rangle) \otimes (|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)
\end{aligned} \tag{4}
$$

Since the perturbation can increase the energy by one unit, we only need to take into account 1st excited states in calculating second order corrections to energy, namely

$$-\sum_{i=1}^{12} \frac{|\langle\psi_i|H^{(1)}|\psi_0\rangle|^2}{E_i - E_0}, \tag{5}$$

and the only non-zero term is for $i = 9$, so we get

$$\Delta E_0 = -\frac{\epsilon^2\hbar^2}{4m\omega^2}. \tag{6}$$

Note that for Eq. (5) to be correct, the states $|\psi_i\rangle$ have to be normalized.

Problem 3

We have

$$p = i\sqrt{\frac{\hbar m\omega}{2}}(a^\dagger - a) \tag{7}$$

and we notice that $\langle k|\frac{\eta p}{m}|n\rangle$ is non-zero only for $k = n \pm 1$. We obtain

$$\begin{aligned} \langle n+1|\frac{\eta p}{m}|n\rangle &= \frac{i\eta}{m}\sqrt{\frac{\hbar m\omega}{2}}\sqrt{n+1}, \\ \langle n-1|\frac{\eta p}{m}|n\rangle &= -\frac{i\eta}{m}\sqrt{\frac{\hbar m\omega}{2}}\sqrt{n}, \end{aligned} \tag{8}$$

so from the first-order time dependent perturbation theory we obtain the amplitudes of the state after time t

$$\begin{aligned} c_n(t) &= 1, \\ c_{n+1}(t) &= -\frac{i}{\hbar}\int_0^t \langle n+1|H^{(1)}|n\rangle e^{i\omega\tau} d\tau = \frac{\eta}{\hbar m}\sqrt{\frac{\hbar m\omega}{2}}\sqrt{n+1}\frac{1}{i\omega}(e^{i\omega t} - 1), \\ c_{n-1}(t) &= -\frac{\eta}{\hbar m}\sqrt{\frac{\hbar m\omega}{2}}\sqrt{n}\frac{1}{-i\omega}(e^{-i\omega t} - 1). \end{aligned} \tag{9}$$

The (normalized) wavefunction after time t is given by

$$|\psi(t)\rangle = |n\rangle + c_{n+1}(t)|n+1\rangle + c_{n-1}(t)|n-1\rangle + \mathcal{O}(\eta^2). \tag{10}$$

Note that this wave function is automatically normalized to $\mathcal{O}(\eta)$ because the time evolution is unitary. The probabilities to transit to state $|n+1\rangle$

and $|n-1\rangle$ are

$$P_{n+1}(t) = |c_{n+1}(t)|^2 \tag{11}$$

$$P_{n-1}(t) = |c_{n-1}(t)|^2, \tag{12}$$

From unitarity, these imply the probablity to stay in $|n\rangle$ is $P_n(t) = 1 - P_{n+1}(t) - P_{n-1}(t)$. The change of energy to leading order in η is given by

$$\begin{aligned} \Delta E &= \langle\psi(T)|H|\psi(T)\rangle - E_n & (13)\\ &= E_{n+1}P_{n+1}(T) + E_{n-1}P_{n-1}(T) + E_nP_n(T) - E_n & (14)\\ &= (E_{n+1} - E_n)P_{n+1}(T) + (E_{n-1} - E_n)P_{n-1}(T) & (15)\\ &= 2\frac{\eta^2}{m}\sin^2\frac{\omega T}{2}. & (16) \end{aligned}$$

The rate is simply $\Delta E/T$.