Physics 125a

Problem Set 3 Solutions

Problem 1

Using $X = \sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger)$ and $P = \frac{i\sqrt{m\hbar\omega}}{\sqrt{2}}(a^\dagger - a)$, along with $a|n\rangle = \sqrt{n}|n-1\rangle$, $a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$, we see that

$$\langle X\rangle = \langle n|X|n\rangle = \sqrt{\frac{\hbar}{2m\omega}}\langle n|(a+a^\dagger)|n\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{n}\langle n|n-1\rangle + \sqrt{n+1}\langle n|n+1\rangle\right) = 0, \quad (1)$$

By orthogonality of $|n\rangle$ states. Similarly,

$$\langle P\rangle = \langle n|P|n\rangle = \frac{i\sqrt{m\hbar\omega}}{\sqrt{2}}\langle n|(a^\dagger - a)|n\rangle = 0. \quad (2)$$

For ΔP, we calculate

$$\begin{aligned}(\Delta P)^2 = \langle n|P^2|n\rangle - \langle n|P|n\rangle^2 &= \frac{m\hbar\omega}{2}\langle n|(a^\dagger - a)(a - a^\dagger)|n\rangle - 0 && (3)\\ &= \frac{m\hbar\omega}{2}\left(\langle n|a^\dagger a|n\rangle - \langle n|a^\dagger|n\rangle - \langle n|aa|n\rangle + \langle n|aa^\dagger|n\rangle\right) && (4)\\ &= \frac{m\hbar\omega}{2}(n - 0 - 0 + n + 1) = m\hbar\omega\left(n+\frac{1}{2}\right) && (5)\end{aligned}$$

Similarly for ΔX,

$$\begin{aligned}(\Delta X)^2 = \langle n|X^2|n\rangle - \langle n|X|n\rangle^2 &= \frac{\hbar}{2m\omega}\langle n|(a^\dagger + a)(a + a^\dagger)|n\rangle - 0 && (6)\\ &= \frac{\hbar}{m\omega}\left(n+\frac{1}{2}\right). && (7)\end{aligned}$$

In summary, $\langle X\rangle = \langle P\rangle = 0$, $\Delta P = \sqrt{m\hbar\omega\left(n+\frac{1}{2}\right)}$ and $\Delta X = \sqrt{\frac{\hbar}{m\omega}\left(n+\frac{1}{2}\right)}$.

Problem 2

(a) Calculate $\langle\psi(0)|X|\psi(0)\rangle$ and $\langle\psi(0)|P|\psi(0)\rangle$.

with $X = \sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger)$ and $P = \frac{i\sqrt{m\hbar\omega}}{\sqrt{2}}(a^\dagger - a)$

$$\langle\psi(0)|X|\psi(0)\rangle = \frac{1}{2}(\langle 0| + \langle 1|)\sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger)(|0\rangle + |1\rangle) = \sqrt{\frac{\hbar}{2m\omega}} \quad (8)$$

$$\langle\psi(0)|P|\psi(0)\rangle = \frac{1}{2}(\langle 0| + \langle 1|)\frac{i\sqrt{m\hbar\omega}}{\sqrt{2}}(a^\dagger - a)(|0\rangle + |1\rangle) = 0 \tag{9}$$

(b) Find $|\psi(t)\rangle$ and use it to compute $\langle\psi(t)|X|\psi(t)\rangle$ and $\langle\psi(t)|P|\psi(t)\rangle$.

$$|\psi(t)\rangle = exp(-iHt)|\psi(0)\rangle = (1/\sqrt{2})exp(-i\frac{\omega t}{2})(|0\rangle + exp(-i\omega t)|1\rangle) \tag{10}$$

$$\langle\psi(t)|X|\psi(t)\rangle = \frac{1}{2}(\langle 0|+exp(i\omega t)\langle 1|)\sqrt{\frac{\hbar}{2m\omega}}(a+a^\dagger)(|0\rangle+exp(-i\omega t)|1\rangle) = \sqrt{\frac{\hbar}{2m\omega}}cos(\omega t) \tag{11}$$

$$\langle\psi(t)|P|\psi(t)\rangle = \frac{1}{2}(\langle 0|+exp(i\omega t)\langle 1|)\frac{i\sqrt{m\hbar\omega}}{\sqrt{2}}(a^\dagger-a)(|0\rangle+exp(-i\omega t)|1\rangle) = -\sqrt{\frac{m\hbar\omega}{2}}sin(\omega t) \tag{12}$$

Problem 3

(a) We have

$$|\lambda\rangle = e^{-\frac{|\lambda|^2}{2}}e^{\lambda a^\dagger}|0\rangle = e^{-\frac{|\lambda|^2}{2}}\sum_{k=0}^{\infty}\frac{\lambda^k(a^\dagger)^k}{k!}|0\rangle, \tag{13}$$

i.e.

$$|\lambda\rangle = e^{-\frac{|\lambda|^2}{2}}\sum_{k=0}^{\infty}\frac{\lambda^k}{\sqrt{k!}}|k\rangle. \tag{14}$$

Then

$$\langle\lambda|\lambda\rangle = e^{-|\lambda|^2}\sum_{k=0}^{\infty}\frac{(\lambda^*)^k\lambda^k}{k!} = 1 \tag{15}$$

as advertised.

(b) From Eq. (14) above

$$\langle n|\lambda\rangle = e^{-\frac{|\lambda|^2}{2}}\frac{\lambda^n}{\sqrt{n!}} \Rightarrow P = e^{-|\lambda|^2}\frac{|\lambda|^{2n}}{n!} \tag{16}$$

so $\nu = |\lambda|^2$.

Problem 4

(c) First we note that $|\lambda\rangle$ is an eigenvector of a, because from Eq. (14) we have

$$a|\lambda\rangle = e^{-\frac{|\lambda|^2}{2}}\sum_{k=0}^{\infty}\frac{\lambda^k}{\sqrt{k!}}a|k\rangle = \lambda e^{-\frac{|\lambda|^2}{2}}\sum_{k=1}^{\infty}\frac{\lambda^{k-1}}{\sqrt{(k-1)!}}|k-1\rangle = \lambda|\lambda\rangle, \tag{17}$$

which also implies

$$\langle\lambda|a^\dagger = \lambda^*\langle\lambda|. \tag{18}$$

Then

$$\langle X\rangle = \sqrt{\frac{\hbar}{2m\omega}}\langle\lambda|(a+a^\dagger)|\lambda\rangle = \sqrt{\frac{\hbar}{2m\omega}}\,(\lambda^*+\lambda) \tag{19}$$

and

$$\langle X^2\rangle = \frac{\hbar}{2m\omega}\langle\lambda|a^2+aa^\dagger+a^\dagger a+(a^\dagger)^2|\lambda\rangle = \frac{\hbar}{2m\omega}\langle\lambda|a^2+1+2a^\dagger a+(a^\dagger)^2|\lambda\rangle = \frac{\hbar}{2m\omega}\left[(\lambda+\lambda^*)^2+1\right], \tag{20}$$

where in the second equality we used $[a,a^\dagger]=1$. Then

$$\Delta X = \sqrt{\langle X^2\rangle - \langle X\rangle^2} = \sqrt{\frac{\hbar}{2m\omega}}. \tag{21}$$

Similarly,

$$\langle P\rangle = i\sqrt{\frac{m\omega\hbar}{2}}\langle\lambda|\left(a^\dagger - a\right)|\lambda\rangle = i\sqrt{\frac{m\omega\hbar}{2}}\,(\lambda^*-\lambda) \tag{22}$$

and

$$\langle P^2\rangle = -\frac{m\omega\hbar}{2}\langle\lambda|(a^\dagger)^2 - a^\dagger a - aa^\dagger + a^2|\lambda\rangle = -\frac{m\omega\hbar}{2}\left[(\lambda^*-\lambda)^2-1\right], \tag{23}$$

so that

$$\Delta P = \sqrt{\frac{m\omega\hbar}{2}}. \tag{24}$$

Then

$$\Delta X\Delta P = \frac{\hbar}{2}. \tag{25}$$

(d) We have

$$|\lambda(t)\rangle = e^{-\frac{|\lambda|^2}{2}}\sum_{k=0}^{\infty} e^{-i\omega t(k+\frac{1}{2})}\frac{\lambda^k}{\sqrt{k!}}|k\rangle \tag{26}$$

so that

$$|\lambda(t)\rangle = e^{-\frac{i\omega t}{2}}|\lambda e^{-i\omega t}\rangle. \tag{27}$$

Thus $\lambda' = \lambda e^{-i\omega t}$.