Ph125: Problem Set 5 Solutions

December 1, 2016

Problem 1

(a) We require a two-particle state which satisfies three properties: it should be invariant under particle exchange, it should be conisistent with finding the particles in states $|\phi\rangle$ and $\psi\rangle$, and it should be normalized. All but the normalization requirement are satisfied by $|\phi\rangle|\psi\rangle + |\psi\rangle|\phi\rangle$. We can ensure the state is normalized by dividing by dividing by an appropriate factor:

$$\frac{|\phi\rangle|\psi\rangle + |\psi\rangle|\phi\rangle}{\sqrt{(\langle\phi\,|\langle\psi\,| + \langle\psi\,|\langle\phi\,||)(|\phi\rangle|\psi\rangle + |\psi\rangle|\phi\rangle)}} \tag{1}$$

$$= \frac{|\phi\rangle|\psi\rangle + |\psi\rangle|\phi\rangle}{\sqrt{2 + 2|\langle\phi\,|\psi\rangle|^2}} \tag{2}$$

(b) No, momentum is not conserved by this system. The reason is that the Hamiltonian does not commute with the momentum operator, and therefore the momentum will change with time in general under time evolution (by Ehrenfest's theorem). To show directly that the Hamiltonian does not commute with the momentum operator, observe that for an arbitrary function $f(x)$:

$$[H,p]f(x) = [p^2/2m + V(x), p]f(x) = [V(x),p]f(x) = [V_0 \sin(2\pi x/a), -\frac{i}{\hbar}\frac{\partial}{\partial x}]f(x) \tag{3}$$

$$= -\frac{iV_0}{\hbar}\left(\sin(2\pi x/a)\frac{\partial f(x)}{\partial x} - \frac{\partial}{\partial x}\left[\sin(2\pi x/a) f(x)\right]\right) \tag{4}$$

$$= -\frac{iV_0}{\hbar}\left(\frac{2\pi}{a}\cos(2\pi x/a) f(x)\right) \neq 0 \tag{5}$$

Problem 2

(a) Substituting the following relations into $|\Psi_{AB}\rangle = \frac{1}{\sqrt{2}}\left(|\hat{\mathbf{z}};\uparrow\rangle_A|\hat{\mathbf{z}};\uparrow\rangle_B + |\hat{\mathbf{z}};\downarrow\rangle_A|\hat{\mathbf{z}};\downarrow\rangle_B\right)$,

$$|\hat{\mathbf{x}};\uparrow\rangle = \tfrac{1}{\sqrt{2}}\left(|\hat{\mathbf{z}};\uparrow\rangle + |\hat{\mathbf{z}};\downarrow\rangle\right) \quad |\hat{\mathbf{x}};\downarrow\rangle = \frac{1}{\sqrt{2}}\left(|\hat{\mathbf{z}};\uparrow\rangle - |\hat{\mathbf{z}};\downarrow\rangle\right) \tag{6}$$

$$|\hat{\mathbf{z}};\uparrow\rangle = \tfrac{1}{\sqrt{2}}\left(|\hat{\mathbf{x}};\uparrow\rangle + |\hat{\mathbf{x}};\downarrow\rangle\right) \quad |\hat{\mathbf{z}};\downarrow\rangle = \frac{1}{\sqrt{2}}\left(|\hat{\mathbf{x}};\uparrow\rangle - |\hat{\mathbf{x}};\downarrow\rangle\right), \tag{7}$$

we obtain,

$$|\Psi_{AB}\rangle = \frac{1}{\sqrt{2}}\left(|\hat{\mathbf{z}};\uparrow\rangle_A|\hat{\mathbf{z}};\uparrow\rangle_B + |\hat{\mathbf{z}};\downarrow\rangle_A|\hat{\mathbf{z}};\downarrow\rangle_B\right) \tag{8}$$

$$= \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\left(|\hat{\mathbf{x}};\uparrow\rangle + |\hat{\mathbf{x}};\downarrow\rangle\right)_A \frac{1}{\sqrt{2}}\left(|\hat{\mathbf{x}};\uparrow\rangle + |\hat{\mathbf{x}};\downarrow\rangle\right)_B + \frac{1}{\sqrt{2}}\left(|\hat{\mathbf{x}};\uparrow\rangle - |\hat{\mathbf{x}};\downarrow\rangle\right)_A \frac{1}{\sqrt{2}}\left(|\hat{\mathbf{x}};\uparrow\rangle - |\hat{\mathbf{x}};\downarrow\rangle\right)_B\right)$$

$$= \frac{1}{\sqrt{2}}\left(|\hat{\mathbf{x}};\uparrow\rangle_A|\hat{\mathbf{x}};\uparrow\rangle_B + |\hat{\mathbf{x}};\downarrow\rangle_A|\hat{\mathbf{x}};\downarrow\rangle_B\right), \tag{9}$$

as required.

Alice and Bob are friends. Bob has the bright idea to use this state to do faster than light communication: he prepares an ensemble with many copies of the state $|\Psi_{AB}\rangle$. and then carefully separates the particles A and B in each pair, keeping particles B in Pasadena and giving particles A to Alice who travels to Pluto. When he wants to send the message Yes he measures his particles spins along the $\mathbf{z}$ direction, which immediately prepares Alice's particles in an ensemble of the states $|\hat{\mathbf{z}};\uparrow\rangle$ and $|\hat{\mathbf{z}};\downarrow\rangle$. For the message No he measures his particles spins along the $\mathbf{x}$ direction, which immediately prepares Alice's particles in an ensemble of the states $|\hat{\mathbf{x}}:\uparrow\rangle$ and $|\hat{\mathbf{x}};\downarrow\rangle$. Alice then measures her spins to see which state Bob been prepared and hence deduce his message. Let us suppose that Alice measures her spins along the $\mathbf{z}$ axis. If Bob has sent the message Yes or No show that Alice will measure spin up half the time and spin down half the time. Similarly if Alice measures her spins along the $\mathbf{x}$ axis. Silly Bob, well at least he isn't stuck on Pluto.

(b) If Bob sends a message Yes, then he measures in the z basis. There is a 0.5 chance the resulting state will be $|\hat{\mathbf{z}};\uparrow\rangle_A|\hat{\mathbf{z}};\uparrow\rangle_B$ which means alice will measure $|\hat{\mathbf{z}};\uparrow\rangle_A$, and a 0.5 chance the resulting state will be $|\hat{\mathbf{z}};\downarrow\rangle_A|\hat{\mathbf{z}};\downarrow\rangle_B$, in which case Alice will measure $|\hat{\mathbf{z}};\downarrow\rangle_A$ (Bob has no control over which).

Alternatively, if Bob sends a message No, then he measures in the x basis. There is a 0.5 chance the resulting state will be $|\hat{\mathbf{x}};\uparrow\rangle_A|\hat{\mathbf{x}};\uparrow\rangle_B$. Given this occurs, Alice's state is $|\hat{\mathbf{x}};\uparrow\rangle_A = \frac{1}{\sqrt{2}}(|\hat{\mathbf{z}};\uparrow\rangle + |\hat{\mathbf{z}};\downarrow\rangle)$, so she will measure $|\hat{\mathbf{z}};\uparrow\rangle_A$ with probability 0.5, and $|\hat{\mathbf{z}};\downarrow\rangle_A$ with probability 0.5. Similarly, there is a 0.5 chance the resulting state will be $|\hat{\mathbf{x}};\downarrow\rangle_A|\hat{\mathbf{x}};\downarrow\rangle_B$. Given this occurs, Aalice's state is $|\hat{\mathbf{x}};\uparrow\rangle_A = \frac{1}{\sqrt{2}}(|\hat{\mathbf{z}};\uparrow\rangle - |\hat{\mathbf{z}};\downarrow\rangle)$, so (as in the other case of Bob's measurement) she will measure $|\hat{\mathbf{z}};\uparrow\rangle_A$ with probability 0.5, and $|\hat{\mathbf{z}};\downarrow\rangle_A$ with probability 0.5. Therefore, irrespective of Bob's outcome, Alice meaures $|\hat{\mathbf{z}};\uparrow\rangle_A$ and $|\hat{\mathbf{z}};\downarrow\rangle_A$ each with probability 0.5.

Hence, irrespective of whether Bob measures in the x or z basis, the outcome statistics of Alice's measurements are the same, so she receives no information about what Bob chose to measure, and therefore does not receive any message, even if they use many entangled pairs.

Problem 3

Show that $\langle J_x\rangle = \langle J_y\rangle = 0$ in the states $|j,m\rangle$ and that in these states

$$\langle J_x^2\rangle = \langle J_y^2\rangle = \frac{1}{2}\hbar^2\left[j(j+1)-m^2\right].$$

transform operators J_x and J_y into J_+ and J_-, we get

$$J_x = \frac{J_+ + J-}{2} \tag{10}$$

$$J_y = \frac{J_+ - J-}{2i} \tag{11}$$

$$J_x^2 = \frac{1}{4}(J_+^2 + J_+J_- + J_-J_+ + J_-^2) \tag{12}$$

$$J_y^2 = -\frac{1}{4}(J_+^2 - J_+J_- - J_-J_+ + J_-^2) \tag{13}$$

then it is obvious that $\langle J_x\rangle = \langle J_y\rangle = 0$, and

$$\langle J_x^2\rangle = \langle J_y^2\rangle = \frac{1}{2}(\langle J^2\rangle - \langle J_z^2\rangle) = \frac{1}{2}\hbar^2\left[j(j+1)-m^2\right] \tag{14}$$

Problem 4

Consider a spinless particle in a state represented by the wave-function

$$\psi(x, y, z) = C(x + y + 2z)\exp(-\gamma r)$$

where $r = \sqrt{x^2 + y^2 + z^2}$.

(a) the angular part of the wave equation $x + y + 2z$ can be expressed in terms of spherical harmonics $Y_1^{\pm}$ and Y_1^0. explicitly we have

$$x + y + 2z \sim (1 + i)Y_1^{-1} + (-1 + i)Y_1^1 + 2\sqrt{2}Y_1^0 \tag{15}$$

since we only have spherical harmonics for $l = 1$, the wave function is an eigenfunction for L^2 with $l = 1$.

(b) the upper index of Y_l^m denotes its eigenvalue of L_z, in unit of $\hbar$

$$\langle L_z \rangle = \hbar \frac{|1 + i|^2 * (-1) + |-1 + i|^2 * 1 + |2\sqrt{2}|^2 * 0}{|1 + i|^2 + |-1 + i|^2 + |2\sqrt{2}|^2} = 0 \tag{16}$$

(c) from the explicit construction of (a) part

$$P(L_z = \hbar) = \frac{|-1 + i|^2}{|1 + i|^2 + |-1 + i|^2 + |2\sqrt{2}|^2} = \frac{1}{6} \tag{17}$$