Physics 125b

Problem Set 5 Solutions

Problem 1

(a) The differential cross section in the Born approximation is given by

$$\frac{d\sigma}{d\Omega} = (2\pi)^4 \mu^2 \hbar^2 |\langle p_f | V | p_i \rangle|^2.$$

Since the potential is a product of two terms, one depending on the position, the other — on the spin, $V = V_{position} V_{spin}$, we obtain

$$\begin{aligned} \frac{d\sigma}{d\Omega} &= (2\pi)^4 \mu^2 \hbar^2 |\langle p_f | V_{position}(\vec{r}) | p_i \rangle|^2 |\langle \chi_f | V_{spin} | \chi_f \rangle|^2 \\ &= \frac{\mu^2}{4\pi^2 \hbar^4} \left| \int d^3\vec{r} e^{-i\vec{q}\cdot\vec{r}} V(\vec{r}) \right|^2 |\langle \chi_f | V_{spin} | \chi_f \rangle|^2 \end{aligned}$$

(b) We have

$$|\chi_i\rangle = |\uparrow\downarrow\rangle = \frac{1}{\sqrt{2}} \left(\frac{|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle}{\sqrt{2}} + \frac{|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle}{\sqrt{2}} \right),$$

where the first state is the triplet state, and the second — the singlet. Also,

$$\vec{s}_a \cdot \vec{s}_b = \frac{1}{2}((\vec{s}_a + \vec{s}_b)^2 - (\vec{s}_a)^2 - (\vec{s}_b)^2).$$

We obtain

$$\begin{aligned} \vec{s}_a \cdot \vec{s}_b \frac{|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle}{\sqrt{2}} &= \frac{\hbar^2}{2} \left(2 - \frac{3}{4} - \frac{3}{4} \right) \frac{|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle}{\sqrt{2}}, \\ \vec{s}_a \cdot \vec{s}_b \frac{|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle}{\sqrt{2}} &= \frac{\hbar^2}{2} \left(0 - \frac{3}{4} - \frac{3}{4} \right) \frac{|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle}{\sqrt{2}}. \end{aligned}$$

Thus,

$$\vec{s}_a \cdot \vec{s}_b |\chi_i\rangle = \frac{\sqrt{5}\hbar^2}{4} \frac{-|\uparrow\downarrow\rangle + 2|\downarrow\uparrow\rangle}{\sqrt{5}} = \frac{\sqrt{5}\hbar^2}{4} |\chi'\rangle,$$

$$\sum_{\chi_f} |\langle \chi_f | V_{spin} | \chi_i \rangle|^2 = \sum_{\chi_f} |\langle \chi_f | \vec{s}_a \cdot \vec{s}_b | \chi_i \rangle|^2 = \frac{5\hbar^4}{16} \sum_{\chi_f} |\langle \chi_f || \chi' \rangle|^2 = \frac{5\hbar^4}{16}.$$

Note that since the summation is over some orthonormal basis (e.g. $|\uparrow\uparrow\rangle$, $|\uparrow\downarrow\rangle$, $|\downarrow\uparrow\rangle$, $|\downarrow\downarrow\rangle$), and $|\chi'\rangle\}$ is normalized, then the sum is equal to 1.

$$\begin{aligned}\frac{d\sigma}{d\Omega} &= \frac{\mu^2}{4\pi^2\hbar^4}\left|\int d^3\vec{r}e^{-i\vec{q}\cdot\vec{r}}V(\vec{r})\right|^2\left(\sum_{|\chi_f\rangle}|\langle\chi_f|V_{spin}|\chi_f\rangle|^2\right)\\ &= \frac{5\mu^2}{64\pi^2}\left|\int d^3\vec{r}e^{-i\vec{q}\cdot\vec{r}}V(\vec{r})\right|^2 = \frac{5\mu^2}{16q^2}\left|\int_{-\infty}^{\infty} dre^{iqr}f(r)r\right|^2\end{aligned}$$

Problem 2

When we measure the final spin of the particle a, then the differential cross section changes. Namely, the only part of the formula from the previous problem which requires some modification is the summation over final states. Now, we need to sum over only the states consistent with the final state, $|\uparrow\uparrow\rangle$, $|\uparrow\downarrow\rangle$, and we obtain

$$\sum_{|\chi_f\rangle=|\uparrow\uparrow\rangle,|\uparrow\uparrow\rangle}|\langle\chi_f|V_{spin}|\chi_i\rangle|^2 = \frac{\hbar^4}{16}.$$

$$\frac{d\sigma}{d\Omega} = \frac{\mu^2}{64\pi^2}\left|\int d^3\vec{r}e^{-i\vec{q}\cdot\vec{r}}V(\vec{r})\right|^2 = \frac{\mu^2}{16q^2}\left|\int_{-\infty}^{\infty} dre^{iqr}f(r)r\right|^2$$

Problem 3

We want to evaluate the Green's function

$$\begin{aligned}G^0(\vec{r}) &= \lim_{\epsilon\to 0}\int \frac{d^3q}{(2\pi)^3}\frac{e^{i\vec{q}\cdot\vec{r}}}{k^2-q^2-i\epsilon}\\ &= \lim_{\epsilon\to 0}\int_0^\infty dq\int_0^\pi d\theta\int_0^{2\pi} d\phi\frac{q^2\sin\theta}{(2\pi)^3}\frac{e^{iqr\cos\theta}}{k^2-q^2-i\epsilon}\\ &= \lim_{\epsilon\to 0}\frac{-i}{4\pi^2 k}\int_{-\infty}^{\infty} dq\frac{q}{k^2-q^2-i\epsilon}e^{-iqr} = \lim_{\epsilon\to 0}\frac{-i}{4\pi^2 k}\int dq f_\epsilon(q)\end{aligned}$$

Note that there are two (simple) poles satisfying $q^2 = k^2 - i\epsilon$, namely $q \simeq k - i\epsilon'$ and $q \simeq -k + i\epsilon'$. We choose the contour to be the upper semicircle.

Using Jordan's lemma and the residue theorem, we obtain

$$G^0(\vec{r}) = \frac{-i}{4\pi^2 k} \lim_{\epsilon\to 0} \int dq f_\epsilon(q) = \frac{-i}{4\pi^2 k} \lim_{\epsilon\to 0} \left(2\pi i \ \mathrm{Res}_{f_\epsilon}(-k+i\epsilon')\right) = -\frac{e^{-ikr}}{4\pi r}.$$

The exponent has minus sign, indicating we have an incoming wave (compared to an outgoing solution obtained in the class).

Problem 4

(a) Since we have bosons, the following commutation relations are satisfied

$$[b^\dagger(\vec{k}_1), b^\dagger(\vec{k}_2)] = 0,$$
$$[b(\vec{k}_1), b^\dagger(\vec{k}_2)] = \delta_{\vec{k}_1,\vec{k}_2}.$$

The state is symmetric, since

$$|\vec{k}_1, \vec{k}_2\rangle = b^\dagger(\vec{k}_1)b^\dagger(\vec{k}_2)|0\rangle = b^\dagger(\vec{k}_2)b^\dagger(\vec{k}_1)|0\rangle = |\vec{k}_2, \vec{k}_1\rangle.$$

(b) Notice that the Hamiltonian can be equivalently written as

$$H = \frac{\hbar^2}{2m}\sum_{\vec{k}} k^2 b^\dagger(\vec{k})b(\vec{k}).$$

We obtain

$$\begin{aligned}
H|\vec{k}_1, \vec{k}_2\rangle &= \frac{\hbar^2}{2m}\sum_{\vec{k}} k^2 b^\dagger(\vec{k})b(\vec{k})b^\dagger(\vec{k}_1)b^\dagger(\vec{k}_2)|0\rangle \\
&= \frac{\hbar^2}{2m}\sum_{\vec{k}} k^2 b^\dagger(\vec{k})(b^\dagger(\vec{k}_1)b(\vec{k}) + \delta_{\vec{k},\vec{k}_1})b^\dagger(\vec{k}_2)|0\rangle \\
&= \frac{\hbar^2}{2m}\left(k_1^2 b^\dagger(\vec{k}_1)(b^\dagger(\vec{k}_2)|0\rangle + \sum_{\vec{k}} k^2 b^\dagger(\vec{k})b^\dagger(\vec{k}_1)(b^\dagger(\vec{k}_2)b(\vec{k}) + \delta_{\vec{k},\vec{k}_2})|0\rangle\right) \\
&= \frac{\hbar^2}{2m}\left(k_1^2 b^\dagger(\vec{k}_1)(b^\dagger(\vec{k}_2)|0\rangle + k_2^2 b^\dagger(\vec{k}_2)(b^\dagger(\vec{k}_1)|0\rangle\right) = \frac{\hbar^2}{2m}(k_1^2 + k_2^2)|\vec{k}_1, \vec{k}_2\rangle
\end{aligned}$$

(c) Using commutation relations from part (a) and transforming the expression in a similar was as in part (b) we obtain

$$\langle \vec{k}_1', \vec{k}_2' | \vec{k}_1, \vec{k}_2\rangle = \langle 0|b^\dagger(\vec{k}_2')b(\vec{k}_1')b^\dagger(\vec{k}_1)b^\dagger(\vec{k}_2)|0\rangle = \delta_{\vec{k}_1',\vec{k}_1}\delta_{\vec{k}_2',\vec{k}_2} + \delta_{\vec{k}_1',\vec{k}_2}\delta_{\vec{k}_2',\vec{k}_1}.$$