Ph125: Problem Set 4 Solutions

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November 16, 2016

Problem 1

(a) The system is in the ground state of old SHO. The ground state is annihilated by its annihilation operator a,

$$a|0_{\text{old}}\rangle = 0. \tag{1}$$

However, it is not true if we replace a by the new annihilation operator b. The old creation/annihilation operator and the new ones can be related by

$$\begin{aligned} X &= (\frac{\hbar}{2m\omega})^{1/2}(a + a^\dagger) = (\frac{\hbar}{2m(4\omega)})^{1/2}(b + b^\dagger) \\ P &= i(\frac{m\omega\hbar}{2})^{1/2}(a^\dagger - a) = i(\frac{m(4\omega)\hbar}{2})^{1/2}(b^\dagger - b). \end{aligned} \tag{2}$$

We find

$$a = \frac{5}{4}b - \frac{3}{4}b^\dagger. \tag{3}$$

The condition in Eq. (1) becomes

$$\frac{5}{2}b|0_{\text{old}}\rangle = \frac{3}{2}b^\dagger|0_{\text{old}}\rangle. \tag{4}$$

The inner product between $|(n+2)_{\text{new}}\rangle$ and $|0_{\text{old}}\rangle$ is

$$\begin{aligned} \langle (n+2)_{\text{new}}|0_{\text{old}}\rangle &= \frac{1}{\sqrt{(n+2)!}}\langle 0_{\text{new}}|b^{n+2}|0_{\text{old}}\rangle \\ &= \frac{3}{5}\frac{1}{\sqrt{(n+2)!}}\langle 0_{\text{new}}|b^{n+1}b^\dagger|0_{\text{old}}\rangle \\ &= \frac{3}{5}\frac{1}{\sqrt{(n+2)!}}\langle 0_{\text{new}}|(n+1)b^n + b^\dagger b^{n+1}|0_{\text{old}}\rangle \\ &= \frac{3(n+1)}{5\sqrt{(n+2)(n+1)}}\langle n_{\text{new}}|0_{\text{old}}\rangle, \end{aligned} \tag{5}$$

where we use Eq. (4) to get the second line, and the remaining lines involve the commutation relation of $[b, b^\dagger] = 1$ only.

Following this relation, the ratio of the probability in eigenstate $|(n+2)_{\text{new}}\rangle$ to $|n_{\text{new}}\rangle$ is

$$R = \left|\frac{\langle (n+2)_{\text{new}}|0_{\text{old}}\rangle}{\langle n_{\text{new}}|0_{\text{old}}\rangle}\right|^2 = \frac{9(n+1)}{25(n+2)}. \tag{6}$$

(b) The wave function of a ground state of a SHO is even, for any frequency. On the other hand, the wave function of first excited state for any SHO is odd. Thus, the probability for the system in $|1_{\text{new}}\rangle$ is zero.

Problem 2

The spectrum of a single particle in a 1D box of length L is

$$E_n = \frac{\hbar^2}{2m}(\frac{n\pi}{L})^2, \tag{7}$$

where n is a positive integer. The total energy of two non-interacting particle is simply the sum of their individual energy.

For $E_{sys} = \hbar^2\pi^2/mL^2$, the only possible choice is $(n_A, n_B) = (1, 1)$. Since the quantum numbers are the same, this is possible only for bosons, not for fermions. The state of the system is

$$|sys\rangle_b = |A\rangle \otimes |B\rangle = |1\rangle \otimes |1\rangle, \tag{8}$$

where the first term of the product specifies the state of particle A while the second is for B.

For $E_{sys} = 5\hbar^2\pi^2/(2mL^2)$, the possible choices are $(n_A, n_B) = (2, 1)$ and $(n_A, n_B) = (1, 2)$. For bosons, we symmetrize it and get

$$|sys\rangle_b = \frac{1}{\sqrt{2}}(|1\rangle \otimes |2\rangle + |2\rangle \otimes |1\rangle); \tag{9}$$

while for fermions, the wave function must be antisymmetric

$$|sys\rangle_f = \frac{1}{\sqrt{2}}(|1\rangle \otimes |2\rangle - |2\rangle \otimes |1\rangle). \tag{10}$$

Problem 3

(a) If particles are distinct, each particle can be in a, b, or c. So there are $3^3 = 27$ possible states.

(b) For bosons, the possible states are

- 1 state: $|a\rangle \otimes |b\rangle \otimes |c\rangle$.
- 6 states: $|a\rangle \otimes |a\rangle \otimes |b\rangle$+ 5 other permutations.
- 3 states: $|a\rangle \otimes |a\rangle \otimes |a\rangle$+ 2 other permutations.

Problem 4

From classical mechanics, the conjugate momentum is given by $p = \partial\mathcal{L}(x, \dot{x})/\partial\dot{x}$. We have

$$p = m\dot{x} + \frac{q}{c}\mathbf{A}. \tag{11}$$

Then, the Hamilitonian follows

$$\begin{aligned} H(p, x) &= p\dot{x} - \mathcal{L}(x, \dot{x}) \\ &= \frac{1}{2}m\dot{x}^2 + q\phi \\ &= \frac{1}{2m}(p - \frac{q}{c}\mathbf{A})^2 + q\phi. \end{aligned} \tag{12}$$

Note that Hamiltonian is written in terms of variable x, p, not $x, \dot{x}$.

(a) Without scalar potential, the magnetic field is given by the $\mathbf{B} = \nabla \times \mathbf{A} = B\hat{z}$.

On the other hand, since only magnetic field is the only measurable, any vector potential which gives the same magnetic field is physically identical. For example, you can verify that the following vector potential gives the same $\mathbf{B}$ field and thus the same physics

$$\mathbf{A} = B(x\hat{y}). \tag{13}$$

(b) From now on, we consider the case without electric field, and therefore $\phi = 0$. In the case, the Lorentz force is always perpendicular to the velocity. So no work is done on the particle and the speed is a constant. It provides the centrifugal force for the circular motion. Using $F = ma$, we have

$$\begin{aligned} m\frac{v^2}{r} &= \frac{qvB}{c}, \\ v &= \frac{qBr}{mc}, \\ \omega_0 &= \frac{v}{r} = \frac{qB}{mc}, \end{aligned} \tag{14}$$

(c) With the $\mathbf{A} = \frac{1}{2}B(-y\hat{x} + x\hat{y})$ and Hamiltonian in Eq. (12), we have

$$\begin{aligned} H &= \frac{1}{2m}[(p_x + \frac{qB}{2c}y)^2 + (p_y - \frac{qB}{2c}x)^2] \\ &= \frac{1}{2m}(p_x^2 + p_y^2) + \frac{1}{2}m(\frac{qB}{2mc})^2(x^2 + y^2) - \frac{qB}{2mc}(xp_y - yp_x) \\ &= \frac{1}{2m}(p_x^2 + p_y^2) + \frac{1}{2}m(\frac{\omega_0}{2})^2(x^2 + y^2) - \frac{\omega_0}{2}\hat{L}_z. \end{aligned} \tag{15}$$

One has to be careful in expanding the square. The $p_{x,y}$ and x, y here are operators. In general, their orders are important and $p_x y \neq y p_x$. However, since they commute so we can treat them like ordinary numbers. So the square can be expanded as normal. The final result shows that it is an isotropic 2D Hamiltonian with an additional term proportional to angular momentum in z-direction.

(d) There are several ways to derive the energy levels.

– From the first line of Eq. (16), we have

$$\begin{aligned} H &= \frac{1}{2m}[(p_x + \frac{qB}{2c}y)^2 + (p_y - \frac{qB}{2c}x)^2] \\ &= \frac{1}{2m}[P^2 + m^2(\frac{qB}{mc})^2 Q^2] \\ &= \frac{1}{2m}P^2 + \frac{1}{2}m\omega_0^2 Q^2, \end{aligned} \tag{16}$$

where $P \equiv p_x + \frac{qB}{2c}y$ and $Q \equiv -\frac{c}{qB}(p_y - \frac{qB}{2c}x) = \frac{1}{2}x - \frac{c}{qB}p_y$. The commutation relation between P and Q is

$$[Q, P] = \frac{1}{2}[x, p_x] - \frac{1}{2}[p_y, y] = i\hbar. \tag{17}$$

Since the commutation relation is the same as the usual x, p_x, one can view Q, P as a new set of genralized coordinate and momentum. This Hamiltonian is a SHO with $\omega = \omega_0$. So the energy levels are

$$E_n = \hbar\omega_0(n + \frac{1}{2}), n \geq 0. \tag{18}$$

– The Hamiltonian in Eq. (16) can be written as

$$\begin{aligned} H &= \frac{1}{2m}(p_x^2 + p_y^2) + \frac{1}{2}m(\frac{\omega_0}{2})^2(x^2 + y^2) - \frac{\omega_0}{2}\hat{L_z}, \\ &= H_{2D} - \frac{\omega_0}{2}\hat{L_z}, \end{aligned} \tag{19}$$

where H_{2D} is the Hamiltonian for 2D isotropic SHO. It's easy to see that it commutes with $\hat{L_z}$

$$[H_{2D}, \hat{L_z}] = 0. \tag{20}$$

So there exists a basis where H_{2D} and $\hat{L_z}$ can be diagonalized simultaneously. Although H_{2D} can be diagonalized using the $|n_x, n_y\rangle$ basis, it doens't commute with L_z. The convenient choice is the one that keeps the rotational invariance. See Exercise 12.3.7. The energy spectrum for H_{2D} is given as

$$E_{2D,(k,m)} = (2k + |m| + 1)\frac{\hbar\omega_0}{2}, k = 0, 1, 2, ... \tag{21}$$

where $m \in \mathcal{Z}$ is the quantum number of angular momentum in z direction. Combining it with the additional term, the overall energy levels are

$$E_{(k,m)} = (2k + |m| - m + 1)\frac{\hbar\omega_0}{2}. \tag{22}$$

Comparing Eq. (22) with Eq. (18), we identify

$$n = k + \frac{|m| - m}{2}, \tag{23}$$

which is non-negative for any m. It is consistent with the result in Eq. (18).

– As we have mentioned, we are free to choose a different **A** as long as the magnetic field is the same. The Hamiltonian with the **A** in Eq. (13) is

$$H = \frac{1}{2m}(p_x^2 + (p_y - \frac{qB}{c}x)^2). \tag{24}$$

Since H only depends on p_y, we use the eigenfunction of p_y as an ansatz for y-dependence

$$\psi(x, y) = f(x)e^{iky}. \tag{25}$$

When H acts on $\psi(x, y)$, we find

$$\begin{aligned} H\psi(x, y) &= [\frac{1}{2m}(p_x^2 + (\hbar k - \frac{qB}{c}x)^2)]f(x)e^{iky} \\ &= [\frac{p_x^2}{2m} + \frac{1}{2}m(\frac{qB}{mc})^2(x - \frac{c}{qB}\hbar k)^2]f(x)e^{iky} \\ &= [\frac{p_x^2}{2m} + \frac{1}{2}m\omega_0^2(x - \frac{c}{qB}\hbar k)^2]f(x)e^{iky}. \end{aligned} \tag{26}$$

The effective Hamiltonian on x is a SHO with frequency ω_0! The momentum in y simply shifts the origin of the oscillator, but doesn't change the energy levels. We have

$$E_n = \hbar\omega_0(n + \frac{1}{2}), n \geq 0. \tag{27}$$

Again, the result is consistent.

Note that although the Hamiltonian and the wave function may be different for different choices of **A**, the energy levels, which is a physical measurable, is the same!