Physics 125b Solutions of Problem Set 2

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Problem 1

For the first order energy sihft of ground state, we need to compute

$$\langle 0|H^{(1)}|0\rangle = \langle 0|\epsilon \mathrm{cos(kX)}|0\rangle \tag{1}$$

use the operator basis with a $a^\dagger$

$$\mathrm{cos(kX)} = \mathrm{cos(k}\sqrt{\frac{\hbar}{2\mathrm{m}\omega}}(\mathrm{a}+\mathrm{a}^\dagger)) = \frac{1}{2}(\mathrm{e}^{\mathrm{ik}\sqrt{\frac{\hbar}{2\mathrm{m}\omega}}(\mathrm{a}+\mathrm{a}^\dagger)} + \mathrm{e}^{-\mathrm{ik}\sqrt{\frac{\hbar}{2\mathrm{m}\omega}}(\mathrm{a}+\mathrm{a}^\dagger)}) \tag{2}$$

use the Baker Hausdorff formula with

$$[\pm ik\sqrt{\frac{\hbar}{2m\omega}}a^\dagger, \pm ik\sqrt{\frac{\hbar}{2m\omega}}a] = k^2\frac{\hbar}{2m\omega} \tag{3}$$

$$\langle 0|H^{(1)}|0\rangle = \frac{\epsilon}{2}e^{-k^2\frac{\hbar}{4m\omega}}\langle 0|e^{ik\sqrt{\frac{\hbar}{2m\omega}}a^\dagger}e^{ik\sqrt{\frac{\hbar}{2m\omega}}a} + e^{-ik\sqrt{\frac{\hbar}{2m\omega}}a^\dagger}e^{-ik\sqrt{\frac{\hbar}{2m\omega}}a}|0\rangle = \epsilon e^{-k^2\frac{\hbar}{4m\omega}} \tag{4}$$

the last equality come from only the first terms in the expansion of the exponentials (which is 1) survive.

Problem 2

From a previous problem set in the S_z basis the angular momentum operators are

$$S_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad S_y = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad S_z = \hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{5}$$

The unperturbed Hamiltonian is

$$H^{(0)} = AS_z^2 \tag{6}$$

and its eigenstates are the eigenstates of S_z^2 and S_z. In the S_z basis they can be written as

$$\psi_{-1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \psi_0 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \psi_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \tag{7}$$

with eigenvalues $A\hbar^2/2$, 0 and $A\hbar^2/2$ respectively, so energy level $A\hbar^2/2$ is degenerate. We use the eigenvectors of the

$$H^{(1)} = B\left(S_x^2 - S_y^2\right) \tag{8}$$

perturbation to lift the degeneracy. They are

$$\psi_{-} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \qquad \psi_0 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \psi_{+} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \tag{9}$$

with energies $-B\hbar^2$, 0 and $B\hbar^2$ respectively, so the states suitable for perturbation theory are ψ_-, ψ_0 and ψ_+ and the energies are

$$\frac{A\hbar^2}{2} - B\hbar^2, \qquad 0, \qquad \frac{A\hbar^2}{2} + B\hbar^2, \tag{10}$$

respectively.

Problem 3

The additional term $H_1 = \gamma|\frac{P}{m}|$ corresponds to the dissipation. It should lower the mechanical energy of the particle when it moves. Therefore, γ should be negative.

As the hint suggested, the operator is less ambiguous if we work in momentum space. The ground state wave function in position space is $\psi(x) = A' \exp[-m\omega x^2/2\hbar]$. Fourier transforming it into momentum space yields

$$\psi(p) = A\, e^{-\frac{p^2}{2m\hbar\omega}} \tag{11}$$

where $A = (\pi m\hbar\omega)^{-1/4}$ is the normalization constant. The shift in ground state energy then follows

$$\begin{aligned}
\Delta E &= \langle 0|H_1|0\rangle \\
&= \int_{-\infty}^{\infty} dp\, \psi^2(p)\, \gamma|\frac{P}{m}| \\
&= 2\frac{\gamma}{m}A^2 \int_{-\infty}^{\infty} dp\, p\, e^{-\frac{p^2}{2m\hbar\omega}} \\
&= \gamma\sqrt{\frac{\hbar\omega}{m\pi}}
\end{aligned} \tag{12}$$