# Physics 125b Solutions of Problem Set 1

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## Problem 1

We will use the notation $|j_1 m_1, j_2 m_2\rangle$ for product state and $|jm\rangle$ after adding the angular momentum as in Shankar. Starting from the largest spin,

$$|\frac{3}{2}\frac{3}{2}\rangle = |\frac{1}{2}\frac{1}{2}, 11\rangle. \tag{1}$$

Then, applying lowering operator $J_-$ on both sides we find

$$|\frac{3}{2}\frac{1}{2}\rangle = \sqrt{\frac{1}{3}}|\frac{1}{2} - \frac{1}{2}, 11\rangle + \sqrt{\frac{2}{3}}|\frac{1}{2}\frac{1}{2}, 10\rangle. \tag{2}$$

For $|\frac{1}{2}\frac{1}{2}\rangle$, it must be linear combination of the same product states on the RHS of the above equation. Using orthogonality between $|\frac{3}{2}\frac{1}{2}\rangle$ and $|\frac{1}{2}\frac{1}{2}\rangle$, we can write down

$$|\frac{1}{2}\frac{1}{2}\rangle = \sqrt{\frac{2}{3}}|\frac{1}{2} - \frac{1}{2}, 11\rangle - \sqrt{\frac{1}{3}}|\frac{1}{2}\frac{1}{2}, 10\rangle. \tag{3}$$

For states with $m < 0$, the Clebsh-Gordon coefficients can be obtained from $m > 0$ using Eq. 15.2.11 in Shankar. We summarize these as

$$|\frac{3}{2} - \frac{1}{2}\rangle = \sqrt{\frac{1}{3}}|\frac{1}{2}\frac{1}{2}, 1-1\rangle + \sqrt{\frac{2}{3}}|\frac{1}{2} - \frac{1}{2}, 10\rangle \tag{4}$$

$$|\frac{1}{2} - \frac{1}{2}\rangle = -\sqrt{\frac{2}{3}}|\frac{1}{2}\frac{1}{2}, 1-1\rangle + \sqrt{\frac{1}{3}}|\frac{1}{2} - \frac{1}{2}, 10\rangle \tag{5}$$

$$|\frac{3}{2} - \frac{3}{2}\rangle = |\frac{1}{2} - \frac{1}{2}, 1-1\rangle. \tag{6}$$

## Problem 2

This problem is very similar from the previous one. We also start with the highest $j, m$

$$|22\rangle = |11, 11\rangle. \tag{7}$$

We can find $j = 2$, $m = 1, 0$ by applying $J_-$ sequentially

$$\begin{aligned} |21\rangle &= \sqrt{\frac{1}{2}}|11, 10\rangle + \sqrt{\frac{1}{2}}|10, 11\rangle && (8) \\ |20\rangle &= \sqrt{\frac{1}{6}}\left(|11, 1-1\rangle + 2|10, 10\rangle + |1-1, 11\rangle\right) && (9) \end{aligned}$$

To find the $j = 1$ sector, we use its orthogonality to $j = 2$ sector

$$\begin{aligned} |11\rangle &= \sqrt{\frac{1}{2}}|11, 10\rangle - \sqrt{\frac{1}{2}}|10, 11\rangle && (10) \\ |10\rangle &= \sqrt{\frac{1}{2}}\left(|11, 1-1\rangle - |1-1, 11\rangle\right) && (11) \end{aligned}$$

The $|00\rangle$ is found by using its orthogonality to $|20\rangle$ and $|10\rangle$

$$|00\rangle = \sqrt{\frac{1}{3}}\left(|11, 1-1\rangle - |10, 10\rangle + |1-1, 11\rangle\right). \tag{12}$$

Finally, the rest of states are completed by Eq. 15.2.11 in Shankar.

$$\begin{aligned} |2-1\rangle &= \sqrt{\frac{1}{2}}|1-1, 10\rangle + \sqrt{\frac{1}{2}}|10, 1-1\rangle && (13) \\ |2-2\rangle &= |1-1, 1-1\rangle && (14) \\ |1-1\rangle &= -\sqrt{\frac{1}{2}}|1-1, 10\rangle + \sqrt{\frac{1}{2}}10, 1-1\rangle && (15) \\ & && (16) \end{aligned}$$

## Problem 3

**(a)** The key here is to use the algebraic commutation relation

$$[L_x, L_y] = i\hbar L_z, \tag{17}$$

and other cyclic permutations of $x, y, z$. We find

$$[L_x, L_z^2] = [L_x, L_z]L_z + L_z[L_x, L_z] = -i\hbar(L_yL_z + L_zL_y)$$
$$[L_y, L_z^2] = [L_y, L_z]L_z + L_z[L_y, L_z] = i\hbar(L_xL_z + L_zL_x).$$

Also, note that $[L_{x,y,z}\, , L^2] = 0$. We already know the Hamiltonian of Hydrogen atom is rotationally invariant so only the last term is relevant. The denominator and the radius $r$ commutes with all $L_{x,y,z}$[1]. The commutators then follow

$$\begin{aligned} [L_x, H] &= i\hbar\epsilon\frac{e^2}{r}\,(L_yL_z + L_zL_y) \\ [L_y, H] &= -i\hbar\epsilon\frac{e^2}{r}\,(L_xL_z + L_zL_x) \\ [L_z, H] &= 0. \end{aligned} \tag{18}$$

Note that we also have $[L^2, H] = 0$.

We see that the new Hamiltonian is no longer rotationally invariant. However, $[L_z, H] = [L^2, H] = 0$. Therefore, they still share the same eigenstates and $|n, l, m\rangle$ is still a good basis.

**(b)** When $H$ acts on spherical harmonics $|l, m\rangle$, the first two terms are identical to Hydrogen atom, but the last term gives

$$-\epsilon\frac{e^2}{r}\frac{m^2}{1 + l(l+1)}. \tag{19}$$

Crucially, this term is still of the form $1/r$ so we can combine with the Columb potential. The effect is simply replacing the electric charge from $e$ to $e(1+\epsilon m^2/(l^2 + l + 1))$. Solving the eigenstate is then the same as Hydrogen atom after the replacement. The energy level of the original Hydrogen atom is $E_n = -me^4/2\hbar^2n^2$. Therefore, the new energy level is

$$E_{n,l,m} = -\frac{me^4}{2\hbar^2n^2}\left(1 + \epsilon\frac{m^2}{1 + l + l^2}\right)^2. \tag{20}$$

**(c)** The energy level for $\epsilon = 3$ is

$$E_{n,l,m} = -\frac{me^4}{2\hbar^2n^2}\left(\frac{l^2 + 3m^2 + l + 1}{1 + l + l^2}\right)^2. \tag{21}$$

[1]The denominator with operator is defined formally as power series. Since $[L_{x,y,z}, L^2] = 0$, the $[L_{x,y,z},\ 1/(\hbar^2 + L^2)] = 0$.

The energy levels of the first couple of states are

$$E_{1,0,0} = -1 \tag{22}$$
$$E_{2,0,0} = -\frac{1}{4} \tag{23}$$
$$E_{2,1,\pm1} = -1 \tag{24}$$
$$E_{3,0,0} = -\frac{1}{9} \tag{25}$$
$$E_{3,1,\pm1} = -\frac{4}{9} \tag{26}$$
$$E_{3,2,\pm2} = -\frac{381}{441} \tag{27}$$

with unit $\frac{me^4}{2\hbar^2}$. The lowest energy level among these states are $|1,0,0\rangle$ and $|2,1,\pm1\rangle$. The second lowest energy level is $|3,2,\pm2\rangle$.

These turn out the be the ground state and first excited states. Consider we fix $n, l$. The lowest energy level happens when $m = \pm l$

$$E_{n,l,\pm l} = -\frac{me^4}{2\hbar^2 n^2}\left(\frac{4l^2+l+1}{1+l+l^2}\right)^2. \tag{28}$$

For given $l$, the lowest energy level is to minimize $n$ as $l+1$. Thus, we can plot $E_l^{\min} = -\frac{me^4}{2\hbar^2}\left(\frac{4l^2+l+1}{(1+l+l^2)(l+1)}\right)^2$. We find it monotonically increases for $l \geq 2$. Thus, the states we listed are sufficient to find ground states and first excited states.

## Problem 4

**(a)** $D^{(1)}[R(\alpha,\beta,\gamma)]$ is just $U[R(\alpha,\beta,\gamma)]$ in the $J_z$ basis. From a previous set we know what the angular momentum operators are in this basis:

$$J_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad J_y = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad J_z = \hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{29}$$

We can exponentiate and multiply the resulting matrices,

$$D^{(1)}[R(\alpha,\beta,\gamma)] = \langle 1,m'|e^{-i\alpha J_z/\hbar}e^{-i\beta J_y/\hbar}e^{-i\gamma J_z/\hbar}|1,m\rangle \tag{30}$$

$$= \begin{pmatrix} e^{-i\alpha} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\alpha} \end{pmatrix} \begin{pmatrix} \frac{\cos(\beta)}{2}+\frac{1}{2} & -\frac{\sin(\beta)}{\sqrt{2}} & \frac{1}{2}-\frac{\cos(\beta)}{2} \\ \frac{\sin(\beta)}{\sqrt{2}} & \cos(\beta) & -\frac{\sin(\beta)}{\sqrt{2}} \\ \frac{1}{2}-\frac{\cos(\beta)}{2} & \frac{\sin(\beta)}{\sqrt{2}} & \frac{\cos(\beta)}{2}+\frac{1}{2} \end{pmatrix} \begin{pmatrix} e^{-i\gamma} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\gamma} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2}e^{-i(\alpha+\gamma)}(\cos(\beta)+1) & -\frac{e^{-i\alpha}\sin(\beta)}{\sqrt{2}} & e^{-i(\alpha-\gamma)}\sin^2\left(\frac{\beta}{2}\right) \\ \frac{e^{-i\gamma}\sin(\beta)}{\sqrt{2}} & \cos(\beta) & -\frac{e^{i\gamma}\sin(\beta)}{\sqrt{2}} \\ e^{i(\alpha-\gamma)}\sin^2\left(\frac{\beta}{2}\right) & \frac{e^{i\alpha}\sin(\beta)}{\sqrt{2}} & \frac{1}{2}e^{i(\alpha+\gamma)}(\cos(\beta)+1) \end{pmatrix}. \tag{31}$$

**(b)** This is direct computation. In this basis

$$\psi = D^{(1)}[R(\alpha,\beta,\gamma)] \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}e^{-i(\alpha+\gamma)}(\cos(\beta)+1) \\ \frac{e^{-i\gamma}\sin(\beta)}{\sqrt{2}} \\ e^{i(\alpha-\gamma)}\sin^2\left(\frac{\beta}{2}\right) \end{pmatrix} \tag{32}$$

so that

$$\langle\psi|J_x|\psi\rangle = \hbar\sin\beta\cos\alpha, \quad \langle\psi|J_y|\psi\rangle = \hbar\sin\beta\sin\alpha, \quad \langle\psi|J_z|\psi\rangle = \hbar\cos\beta, \tag{33}$$

as advertised.

**(c)** The condition is

$$\begin{pmatrix} \frac{1}{2}e^{-i(\alpha+\gamma)}(\cos(\beta)+1) \\ \frac{e^{-i\gamma}\sin(\beta)}{\sqrt{2}} \\ e^{i(\alpha-\gamma)}\sin^2\left(\frac{\beta}{2}\right) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \tag{34}$$

which cannot be satisfied.