Physics 127C – Homework Set 1 (due April 18)

Problem 1: Gaussian integrals

All perturbative calculations use some "free theory" (Gaussian theory) as a starting point. We will also use Gaussian integrals in the spin wave theory later in the course. Prove and (own!) the following main rules of Gaussian integration.

a) Let $\hat{A} = (A)_{ij}$ be a real symmetric positive-definite $N \times N$ matrix, $\mathbf{x} = \text{vector}(x_1, x_2, ..., x_N)$ – a variable vector, $\mathbf{b} = \text{vector}(b_1, b_2, ..., b_N)$ – a constant vector; all integrals are over $[-\infty, +\infty]$. Show that

$$\int d\mathbf{x} \, \exp\left[-\frac{1}{2}\mathbf{x}^t A\mathbf{x} + \mathbf{b}^t\mathbf{x}\right] = \frac{(2\pi)^{N/2}}{\sqrt{\det A}} \exp\left[\frac{1}{2}\mathbf{b}^t A^{-1}\mathbf{b}\right]. \tag{1}$$

Hint: There are many ways to proceed, e.g., by diagonalizing the matrix $\hat{A}$ and using an elementary Gaussian integral.

Remark: The above formula also holds for complex symmetric $\hat{A}$ such that $Re(\hat{A})$ is positive definite, and arbitrary complex vector $\mathbf{b}$ (no conjugation!).

b) Gaussian averages are defined by

$$\langle O(\mathbf{x})\rangle \equiv \frac{\int d\mathbf{x}\, O(\mathbf{x}) \exp\left[-\frac{1}{2}\mathbf{x}^t A\mathbf{x}\right]}{\int d\mathbf{x} \exp\left[-\frac{1}{2}\mathbf{x}^t A\mathbf{x}\right]}. \tag{2}$$

By differentiating the formula in a) with respect to formal parameters b, prove the following results for basic averages:

$$\langle x_i x_j\rangle = (A^{-1})_{ij}; \tag{3}$$

$$\langle x_i x_j x_k x_l\rangle = \langle x_i x_j\rangle\langle x_k x_l\rangle + \langle x_i x_k\rangle\langle x_j x_l\rangle + \langle x_i x_l\rangle\langle x_j x_k\rangle. \tag{4}$$

The last formula is the simplest example of Wick's theorem: the expectation value is given by summing over all possible pairwise "contractions". Thus, when working with Gaussian integrals, we need to calculate $\langle x_i x_j\rangle$ once and then any other average follows by Wick's theorem.

c) Show another very useful formula,

$$\langle \exp[L(\mathbf{x})]\rangle = \exp\left[\frac{1}{2}\langle L(\mathbf{x})^2\rangle\right] , \tag{5}$$

where $L(\mathbf{x}) = \sum_i b_i x_i$ is an arbitrary linear function of Gaussian-distributed variables. Note again that in the final expression we need to know only the basic averages $\langle x_i x_j\rangle$.

Problem 3: Euclidean path integral for quantum XY rotors

Here we consider a quantum many-body system whose Euclidean path integral gives essentially the classical XY model. This problem will guide you through the corresponding material in the lecture notes (see also Sondhi *et al.* RMP article) that we did not have time to do in class; try the problem on your own first and check the notes when stuck.

First, let us define a quantum rotor. It is specified by a canonical pair $\hat{\phi}, \hat{n}$, where $\hat{\phi}$ is a 2π-periodic variable ("angle"), and $\hat{n}$ is the conjugate momentum variable (also called "number operator" for reasons to become clear below). The commutation relations are

$$[\hat{\phi}, \hat{n}] = i \tag{6}$$

For a explicit realization of the quantum rotor, one can consider a Hilbert space of complex-valued functions $\Psi(\phi)$ of variable ϕ that are 2π-periodic, i.e., $\Psi(\phi + 2\pi) = \Psi(\phi)$. One can think of them as Schrondiger wavefunctions of "position" variable ϕ residing on a unit circle. The conjugate momentum operator is then

$$\hat{n} = -i\frac{\partial}{\partial \phi}, \tag{7}$$

and the commutation relation of $\hat{\phi}$ and $\hat{n}$ follows easily.

- Find the eigenfunctions and eigenvalues of $\hat{n}$,

  $$\hat{n}\Psi_m(\phi) = m\Psi_m(\phi); \;\; \hat{n}|m\rangle = m|m\rangle. \tag{8}$$

  Note that the 2π-periodicity makes the eigenvalues of $\hat{n}$ discrete (here, integer-valued, hence the name "number operator"). The first equation is in the Schrondiger wavefunction formulation, while the second equation is in the more abstract Dirac "ket" formulation. Recalling that the Schrondiger wavefunction gives coefficients of the expansion in the position ϕ basis,

  $$|m\rangle = \int_\phi |\phi\rangle\langle\phi|m\rangle = \int_\phi \Psi_m(\phi)|\phi\rangle \tag{9}$$

  obtain the overlaps $\langle\phi|m\rangle$. (Here and below, I am being sloppy about the normalization constants, but you can be more careful if you like.)

- Now consider a Hamiltonian for a single rotor

  $$\hat{H} = \frac{u}{2}\hat{n}^2 \tag{10}$$

  and find the eigenspectrum. What are the ground state and first excited states? What is the excitation gap $\Delta E = E_1 - E_0$ from the ground state to the first excited state?

- Not develop the imaginary time path integral for the Hamiltonian in Eq. (10), working in the ϕ basis. By inserting a decomposition of unity in the basis of $\hat{n}$, argue that

  $$\langle\phi(\tau+\delta\tau)|e^{-\frac{\delta\tau u \hat{n}^2}{2}} = \sum_{m=\infty}^{\infty} e^{-\frac{\delta\tau u}{2}m^2} e^{im[\phi(\tau+\delta\tau)-\phi(\tau)]}. \tag{11}$$

- For small $\delta\tau$, the above expression is not convenient since the sum contains many rapidly oscillating terms of roughly the same magnitude. Such sums can be handled using Poisson resummation formula

  $$\sum_{m=-\infty}^{\infty} e^{-\frac{1}{2}am^2+im\theta} = \sqrt{\frac{2\pi}{a}} \sum_{p=-\infty}^{\infty} e^{-\frac{1}{2a}(\theta+2\pi p)^2}. \tag{12}$$

  Prove this formula by starting with the following identity,

  $$\sum_m \delta(x-m) = \sum_{p=-\infty}^{\infty} e^{i2\pi px}, \tag{13}$$

  where the lhs is the periodic (delta) function with period and the rhs is the corresponding Fourier series.

- For small $\delta\tau$, argue that we can approximate Eq. (11) by

  $$\text{const x } e^{\frac{1}{\delta\tau u}}\cos[\phi(\tau+\delta\tau)-\phi(\tau)]. \tag{14}$$

  Collecting now contributions from different time slices, we obtain the one-dimensional XY model with coupling $K_\tau = \frac{1}{\delta\tau u}$.

- Now you are ready to write the Euclidean path integral for a system of coupled rotors, say residing on the sites of some d-dimensional lattice (labelled r), with the interacting Hamiltonian
$$\hat{H} = \frac{u}{2}\sum_r \hat{n}_r^2 - J\sum_{\langle rr'\rangle}\cos(\phi_r - \phi_{r'}). \qquad (15)$$
Show that the result is the classical XY model in $d+1$ dimension and obtain the couplings in the spacial and temporal directions.

Problem 1: Path integral for the quantum Ising model in the σ^x basis

Here we develop path integral for the quantum Ising model

$$\hat{H} = -J\sum_{\langle ij\rangle}\sigma_i^z\sigma_j^z - \Gamma\sum_i \sigma_i^x \qquad (16)$$

working in the basis of σ^x.

a) Consider first the model with $J = 0$. Even though it can be solved exactly, let us still proceed by expressing the partition function $Z = \mathrm{Tr}e^{-\beta\hat{H}}$ as a path integral working in the basis of σ^x. To visualize different contributions, draw a line segment between i, τ and $i, \tau+\delta\tau$ whenever $S_i^x(\tau) = -1$. What statistical weight should be associated with such a segment relative to the case with no segment [i.e., $S_i^x(\tau) = +1$]? Note that for $J = 0$ we get lines running straight in the τ direction.

b) Consider now $J \neq 0$ and examine $e^{\delta\tau J\sigma_i^z\sigma_j^z} \approx 1+\delta\tau J\sigma_i^z\sigma_j^z$ sandwiched between $\langle S_i^x(\tau+\delta\tau), S_j^x(\tau+\delta\tau)|\ldots|S_i^x(\tau), S_j^x(\tau)\rangle$. What happens to the lines from a) due to the first and second terms? To figure this out, consider cases *1)* when there are no lines entering i, τ and j, τ from $\tau - \delta\tau$; *2)* when there is one line entering; *3)* and when there are two lines entering. For the contributions from the second term, we can maintain the continuity of lines by placing a segment connecting i, τ and j, τ. What is the statistical weight associated with such a segment relative to the case with no segment?

The final result is a sum over worldlines of "$S^x = -1$ particles" with specific weights associated with temporal and spatial segments that you derived.

c) Consider now the usual formulation of the quantum Ising model in the S^z variables, with the spatial coupling $K_{spat} = J\delta\tau$ and temporal coupling $K_\tau = \mathrm{atanh}(e^{-2\Gamma\delta\tau})$. Extend the high-temperature expansion analysis from the class to this space-time anisotropic case and compare with the described path integral in the S^x variables.