The Feynman Path Integral Formulation of Q.M.

We won't use it much. But it's important to know.

1) Principle of least action

Start with a particle of mass m moving in one dimension with coordinate x under the influence of a potential $V(x)$. According to Newton's Law

$$m \frac{d^2x}{dt^2} = -\frac{dV}{dx}$$

In classical physics we have potential energy $V(x)$ and kinetic energy

$$T = \frac{1}{2} m \dot{x}^2$$

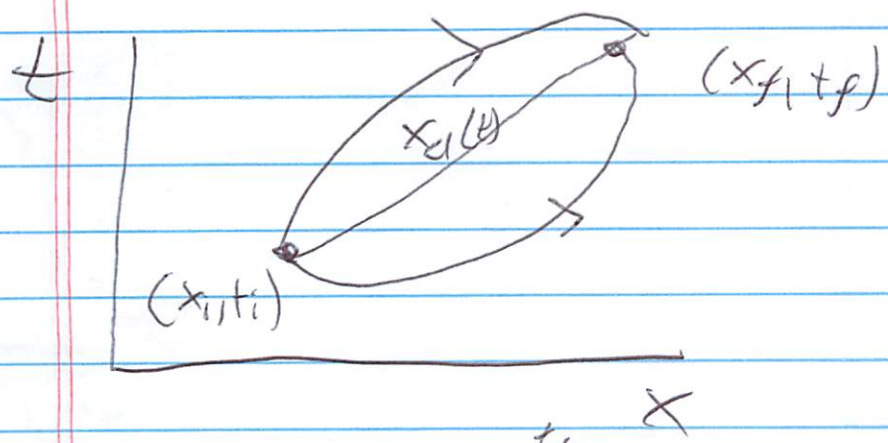
Total energy is $E = T + V$. We can also use the Lagrangian:

$$L(x, \dot{x}) = T - V$$

If we consider paths $x(t)$ that join (x_i, t_i) with (x_f, t_f), the

classical path that satisfies the dynamical equations of motion is the one that minimizes the <u>action</u>

$$S[x(t)] = \int_{t_i}^{t_f} dt\, L(x, \dot{x})$$



$$\delta S[x(t)] = \int_{t_i}^{t_f} \left(\frac{\partial L}{\partial x} \delta x + \frac{\partial L}{\partial \dot{x}} \delta \dot{x} \right) dt$$

$$= \int_{t_i}^{t_f} \left(\frac{\partial L}{\partial x} \delta x + \frac{\partial L}{\partial \dot{x}} \frac{d}{dt} \delta x \right) dt$$

$$= \underbrace{\frac{\partial L}{\partial \dot{x}} \delta x \Big|_{t_i}^{t_f}}_{0} + \int_{t_i}^{t_f} \left(\frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} \right) \delta x \, dt$$

holds for arbitrary δx s.t. $\delta x(t_i, t_f) = 0$

So

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}} \right) = 0$$

Euler-Lagrange eq.

$$\frac{\partial L}{\partial x} = -\frac{dV}{dx}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = m\ddot{x}, \qquad m\ddot{x} = -\frac{dV}{dx} \qquad \text{ok.}$$

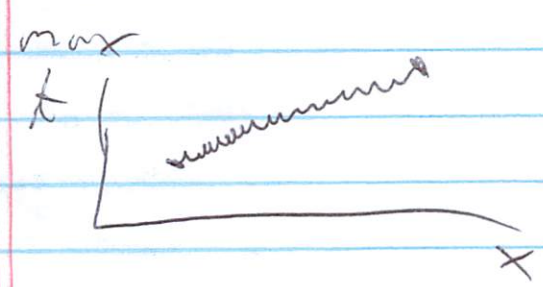
Easily generalizes to many particles in 3 dim:

$$T = \sum_{i=1}^{3n} \frac{1}{2} m_i \dot{x}_i^2, \quad V = V(x_1, \ldots, x_{3n})$$

$$S = \int_{t_i}^{t_f} dt\, L(x_i, \dot{x}_i), \quad \delta x_i(t_i, f_i) = 0$$

$$\delta S = 0 \Rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} = \frac{\partial L}{\partial x_i}$$

The classical path is a stationary point but it's max or min? Certainly not



Wiggly path has almost same V as classical path, but much higher kinetic energy T. So classical path not a

maximum of S.

It ends up it's not always a min either.
Consider

$$L = \frac{1}{2}\dot{x}^2 - \frac{1}{2}x^2$$

classical path with $x = x_0$, @ $t_i = 0$

and $t_f = 2\pi$ is

$$x(t) = x_0 \cos(t)$$

It satisfies Euler-Lagrange Eqs:

$$0 = \frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = -x - \ddot{x} = 0$$

Parametrize a class of neighboring paths as

$$x(t) = (1-\alpha)x_0 \cos t + \alpha x_0$$

For these paths

$$-x + \ddot{x} = -(1-\alpha)x_0 \cos t + \alpha x_0$$
$$+ (1-\alpha)x_0 \cos t = \alpha x_0 \neq 0 \quad \text{for } \alpha \neq 0$$

So α characterizes how far away from classical path

$$S[\alpha] = \int_0^{2\pi} dt \; \frac{1}{2}\left[(1-\alpha)x_0 \sin t\right]^2 - \frac{1}{2}\left[(1-\alpha)x_0 \cos t + x_0\alpha\right]^2$$

$$= \int_0^{2\pi} dt \left[\frac{1}{2}(1-\alpha)^2 x_0^2 \sin^2 t - \frac{1}{2}(1-\alpha)^2 x_0^2 \cos^2 t - \alpha(1-\alpha)x_0^2 \cos^2 t - \frac{1}{2}\alpha^2 x_0^2\right] = -\frac{1}{2}\alpha^2 x_0^2 2\pi$$

So for $\alpha \neq 0$, this non classical path has lower energy. But difference $\propto \alpha^2$, so $\alpha = 0$ is a stationary point

Feynman's path integral

$$\langle x_f | U(t_f, t_i) | x_i \rangle$$

$$\equiv \sum_{\substack{\text{paths } x(t) \\ x(t_i) = x_i \\ x(t_f) = x_f}} e^{iS[x(t)]/\hbar}$$

For very small $\hbar$ the point of stationary phase (which is classical) path

dominates.

This is a pretty simplified type of integral. But I can at least show you how stationary phase works for one-dimensional integrals:

$$\varphi = \int_a^b dx\, e^{iS(x)/\hbar} \qquad \left(= \cos\left[\frac{S(x)}{\hbar}\right] + i \sin\left[\frac{S(x)}{\hbar}\right] \right)$$

Where $\hbar$ is very small. We suppose in (a,b) there is one stationary point x_0 where

$$S'(x_0) = 0$$

Expect if we are not in vinicity of stationary point, wild fluctuations in integrand averages to zero. So expand around stationary point:

$$S(x) = S(x_0) + \frac{1}{2} S''(x_0)(x-x_0)^2 + \dots$$

$$\varphi = \exp\left[i \frac{S(x_0)}{\hbar}\right] \int_a^b dx \exp\left[i \frac{(x-x_0)^2}{2\hbar} S''(x_0) + \frac{i}{6\hbar} S'''(x_0)(x-x_0)^3 + \dots\right]$$

positive

$$y = x - x_0$$

$$\varphi = \exp\left[\frac{i\,S(x_0)}{\hbar}\right] \int_{a-x_0}^{b-x_0} dy \, \exp\left[i y^2 \frac{S''(x_0)}{2\hbar}\right.$$

negative

$$\left. + \frac{i}{6\hbar} S'''(x_0)\, y^3 + \cdots \right]$$

$$z \equiv \sqrt{\frac{S''(x_0)}{2\hbar}}\; y$$

$$\varphi = \exp\left[\frac{i\,S(x_0)}{\hbar}\right] \int_{(a-x_0)\sqrt{\frac{S''(x_0)}{2\hbar}}}^{(b-x_0)\sqrt{\frac{S''(x_0)}{2\hbar}}} dz \, \sqrt{\frac{2\hbar}{S''(x_0)}}$$

$$\cdot \exp\left[i z^2 + \underbrace{\frac{i}{6\hbar} S'''(x_0)\, z^3 \left(\frac{2\hbar}{S''(x_0)}\right)^{3/2} + \cdots}_{O(\hbar)} \right.$$

neglect for very small $\hbar$.

Very small $\hbar$ approx $-\infty$ to ∞;

$$\varphi \simeq \exp\left[\frac{i\,S(x_0)}{\hbar}\right]\sqrt{\frac{2\hbar}{S''(x_0)}}\int_{-\infty}^{\infty} dt\, \exp\left[i z^2\right]$$

But $\int_{-\infty}^{\infty} \sin^2(x)\,dx = \int_{-\infty}^{\infty} \cos^2(x)\,dx = \sqrt{\frac{\pi}{2}}$

$$\varphi \simeq \exp\left[\frac{i\,S(x_0)}{\hbar}\right]\sqrt{\frac{2\pi\hbar}{S''(x_0)}}\left(\frac{1+i}{\sqrt{2}}\right)$$

$$= \exp\left[\frac{i\,S(x_0)}{\hbar}\right]\sqrt{\frac{2\pi\hbar\, i}{S''(x_0)}}$$

Lets go back to Q.M. just keep exponential.

$$\langle x|\,U(t,0)\,|x'\rangle \sim \exp\left[\frac{i\,S(x_{cl})}{\hbar}\right]$$

Free particle: $v = \frac{(x-x')}{t}$

$$T = \frac{1}{2} m \frac{(x-x')^2}{t^2}, \quad S_{cl} = \frac{1}{2} m \frac{(x-x')^2}{t}$$

$$\langle x|\,U(t,0)\,|x'\rangle \simeq \exp\left[\frac{i m (x-x')^2}{2\hbar t}\right]$$

which agrees with what we found using usual Q.M.

Just how big is S compared with $\hbar$?
Suppose particle with mass 20 grams and velocity 100 cm/sec.

$$L = T = \frac{1}{2} m v^2 = \frac{1}{2}(20)\, 10^4 \text{ gm} \cdot \text{cm}^2/\text{sec}^2$$

$$\approx 10^5 \text{ gm} \cdot \text{cm}^2/\text{sec}^2$$

If it travels for 100 sec

$S = 10^7$. But $\hbar \approx 1.05 \times 10^{-27}$

$$\frac{S}{\hbar} \approx \frac{10^7}{10^{-27}} \approx 10^{-34}$$

Action along classical huge compared with $\hbar$.