Equilibration and Unitary k-Designs

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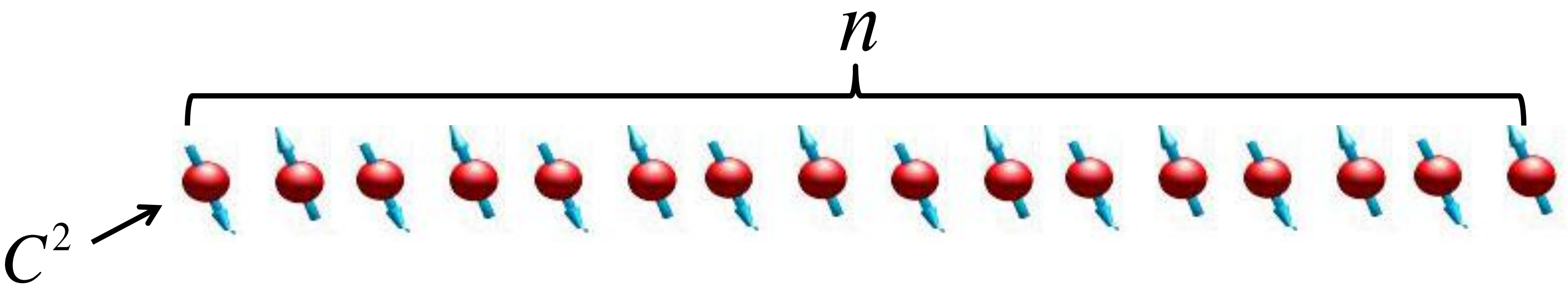
Joint work with
Aram Harrow and Michal Horodecki
arXiv:1208.0692

IMS, September 2013

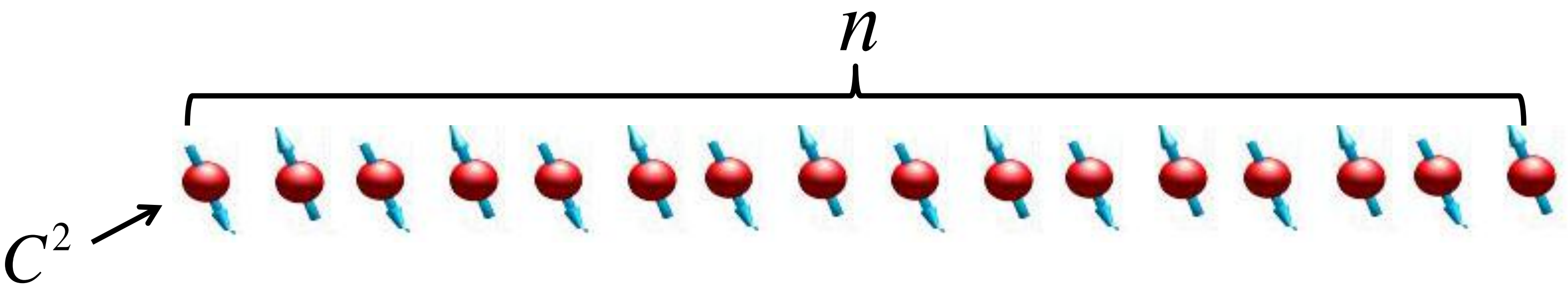
Dynamical Equilibration



State at time *t*: $|\psi_t\rangle = U(t)|0\rangle^{\otimes n}$

$$U(t) = \mathcal{T}e^{i\int_0^t H(t')dt'}$$

Dynamical Equilibration



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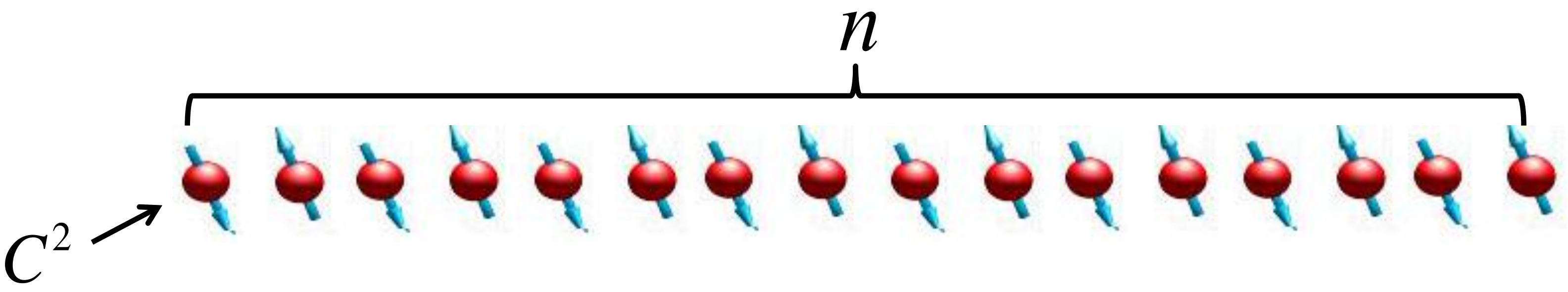
Will $|\psi(t)\rangle$ equilibrate?

I.e. for most *t* $|\psi(t)\rangle\langle\psi(t)| \approx \rho_0$?

Dynamical Equilibration



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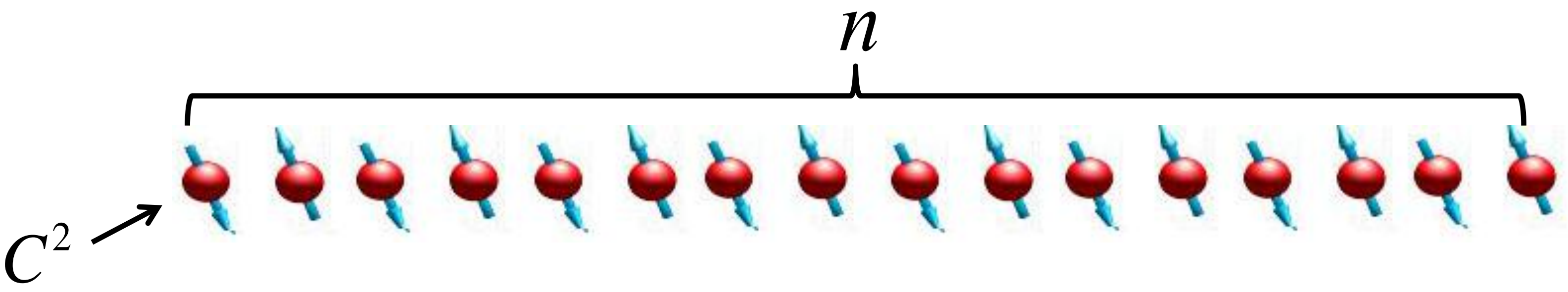
$$U(t) = \mathcal{T} e^{i\int_0^t H(t')dt'}$$

Will $|\psi(t)\rangle$ equilibrate?

I.e. for most *t* $|\psi(t)\rangle\langle\psi(t)| \approx \rho_0$? NO!

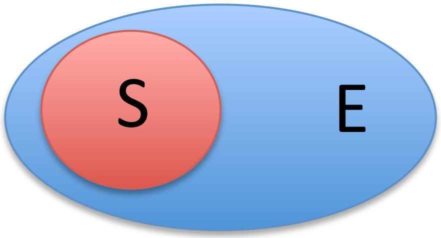
Dynamical Equilibration



How about relative to particular kind of measurements?

- "macroscopic" measurements (von Neumann '29)
- local measurements
- local measurements relative to an external observer (Lidia's talk)
- Low-complexity measurements
  (i.e. measurements that require time much less than *t*)

Equilibration is generic



(Linden, Popescu, Short, Winter '08)

Almost any Hamiltonian *H* with equilibrate:

$$\mathbb{E}_{t\geq 0}\|\rho_S(t)-\overline{\rho_S}\|_1 \approx 0$$

with $\rho_S(t) = \mathrm{tr}_{\setminus S}(e^{itH}|0^n\rangle\langle 0^n|e^{-itH})$

and $\overline{\rho_S} = \lim_{T\to\infty} \mathbb{E}_{t\leq T}\rho_S(t)$

Time Scale of Equilibration

The previous approach only gives bounds exponentially small in the number of particles

Can we prove *fast* equilibration is generic?

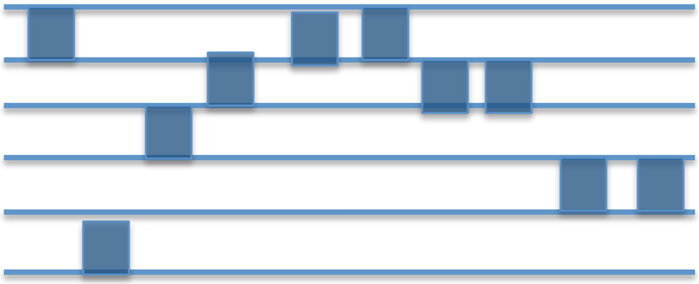
For particular cases better bounds are known
E.g. (Cramer et al ‘08), (Banuls, Cirac, Hastings ‘10), …

This talk:
Generic local dynamics leads to rapid equilibration

Caveat: time-dependent Hamiltonians…

Random Quantum Circuits

Local Random Circuit: in each step an index *i* in {1, … ,n} is chosen uniformly at random and a two-qubit Haar unitary is applied to qubits i e i+1



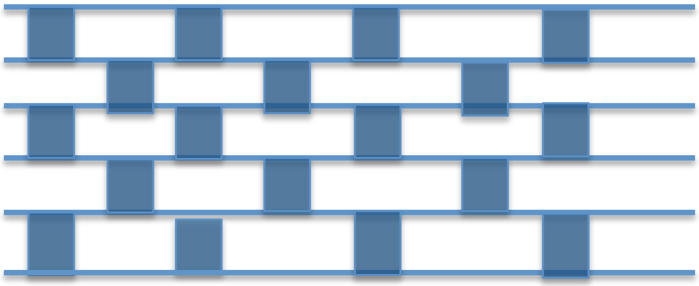
Random Walk in **U**(2^n)

(Another example: Kac's random walk – toy model Boltzmann gas)

Introduced in (Hayden and Preskill '07) as a toy model for the dynamics of a black hole

Parallel Random Quantum Circuits

Parallel Local Random Circuit: in each step $n/2$ independent Haar two-qubit gates are applied to either ((1, 2), (3, 4), …,(n-1,n)) or ((2, 3), (4, 5), …,(n-2,n-1))



Discrete version of $U(t) = \mathcal{T} e^{i \int_0^t H(t')dt'}$

with random $H(t) = H_{12}(t) + H_{23}(t) + \ldots + H_{nn-1}(t)$

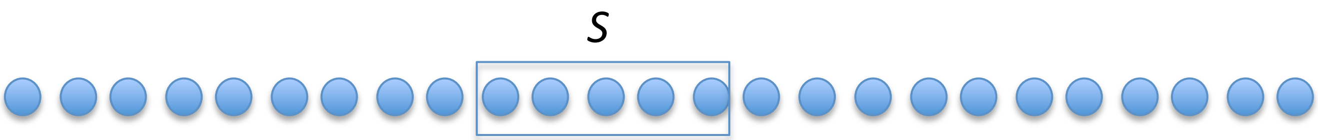
Equilibration for Random Circuits

Thm Let RC_t := { U : U = $U_1 \ldots U_t$} be the set of all circuits of length *t*

1. For every region *S*, $\mathbb{E}_{U \in \mathrm{RC}_t} \|\rho_S(t) - \tau\|_1 \leq \varepsilon$

   for $t \geq O(\log(1/\varepsilon)\mathrm{size}(S))$

   with $\rho_S(U) := \mathrm{tr}_{\setminus S}(U|0\rangle\langle 0|^{\otimes n} U^\dagger)$



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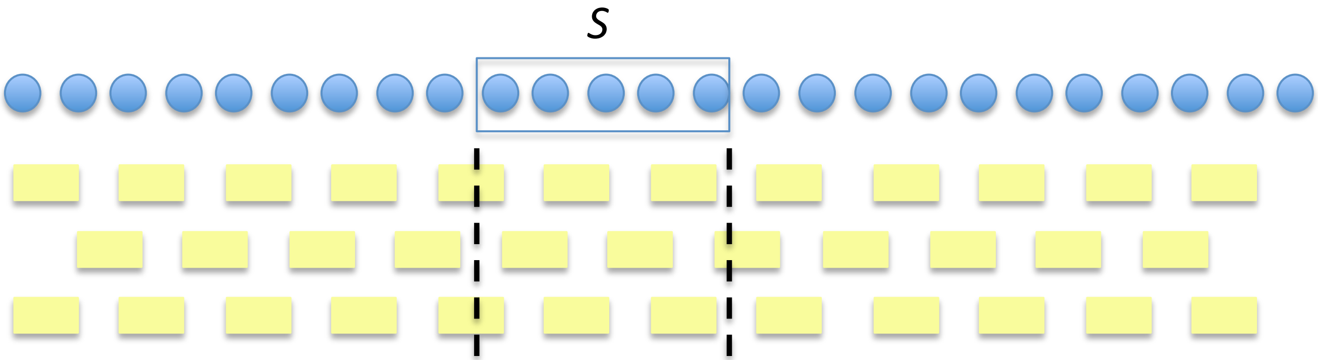


The result matches the speed-of-sound propagation bound

Equilibration for Random Circuits

Thm Let RC_t := { U : U = $U_1 \ldots U_t$} be the set of all circuits of length *t*

2. Let M_k := { M : 0 ≤ M ≤ id, M has gate complexity k }.
For every $t \geq O(k^6 \log(1/\varepsilon))$

$$\mathbb{E}_{U \in RC_t} \max_{N \in M_k} \left| \langle 0^n | U^\dagger N U | 0^n \rangle - \mathrm{tr}(N)/2^n \right| \leq \varepsilon$$

Equilibration for arbitrary measurements of *low* complexity

Fails for t = k, as one can undo the evolution *U*

Warm-up: Equilibration for Haar Random Unitaries

(Page '93)

Haar measure

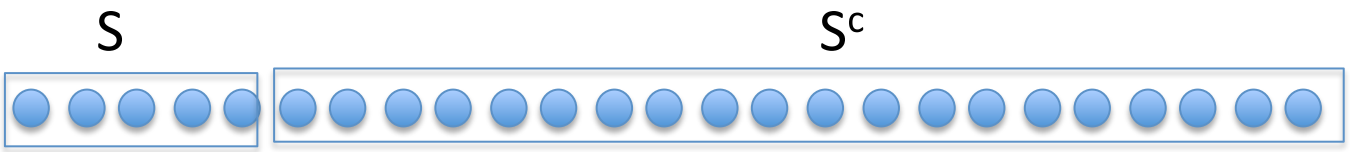
Let $|\psi\rangle = U|0\rangle^{\otimes n}$ and $\rho_S(U) = \mathrm{tr}_{S^c}(U|0\rangle\langle 0|^{\otimes n}U^\dagger)$

We have $\|\rho_S(U) - \tau\|_1 \leq \dim(S)^{1/2}\|\rho_S(U) - \tau\|_2$

But $\mathbb{E}_U\|\rho_S(U) - \tau\|_2^2 = \frac{\dim(S)}{\dim(S^c)}$

only second moments needed

So $\|\rho_S(U) - \tau\|_1 \leq \varepsilon$ for $\log(\dim(S^c)) \geq 2\log(\dim(S)/\varepsilon)$



Unitary *k*-designs

Def. An ensemble of unitaries {μ(dU), U} in **U**(d) is an *ε*-approximate unitary *k*-design if for every monomial $M = U_{p1,\, q1} \ldots U_{pk,\, qk} U^*_{r1,\, s1} \ldots U^*_{rk,\, sk},$

$$|E_\mu(M(U)) - E_{Haar}(M(U))| \leq \varepsilon$$

First *k* moments are close to the Haar measure

Equivalent to (≈)

$$\left\| \int \mu(dU) U^{\otimes k} \otimes \overline{U}^{\otimes k} - \int \mu_{\text{Haar}}(dU) U^{\otimes k} \otimes \overline{U}^{\otimes k} \right\|_\infty \leq \varepsilon$$

Unitary *k*-designs

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$$|E_\mu(M(U)) - E_{Haar}(M(U))| \leq \varepsilon$$

Natural quantum generalization of *k*-wise independent distributions

Many applications in quantum information theory: encoding for quantum communication (2-design), generic speed-ups (3-design), efficient tomography (4-design), …

Unitary *k*-designs and equilibration

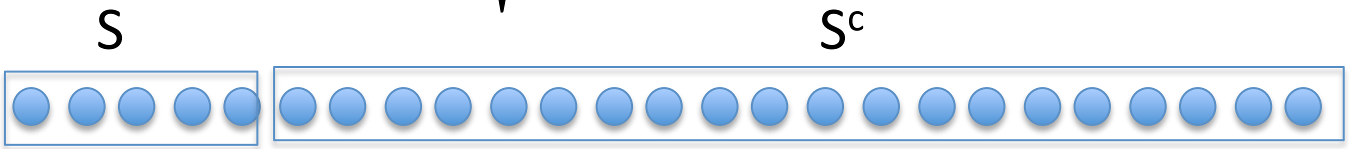
From δ-approx 2-design

Let $|\psi\rangle = U|0\rangle^{\otimes n}$ and $\rho_S(U) = \mathrm{tr}_{S^c}(U|0\rangle\langle 0|^{\otimes n} U^\dagger)$

We have $\|\rho_S(U) - \tau\|_1 \leq \dim(S)^{1/2} \|\rho_S(U) - \tau\|_2$

But $\mathbb{E}_U \|\rho_S(U) - \tau\|_2^2 = \dfrac{\dim(S)}{\dim(S^c)} + \delta$

So $\|\rho_S(U) - \tau\|_1 \leq \sqrt{\dfrac{\dim(S)}{\dim(S^c)} + \delta \dim(S)}$



Equilibration for Random Circuits

Thm Let RC_t := { U : U = $U_1 \ldots U_t$} be the set of all circuits of length *t*

2. Let M_k := { M : 0 ≤ M ≤ id, M has gate complexity k }.

For every $t \geq O(k^6 \log(1/\varepsilon))$

$$\mathbb{E}_{U \in RC_t} \max_{N \in M_k} \left| \langle 0^n | U^\dagger N U | 0^n \rangle - \mathrm{tr}(N)/2^n \right| \leq \varepsilon$$

Proof follows by looking at higher moments to obtain a good concentration bound on $<0^n|U^T N U|0^n>$ and union bound over set M_k.

Requires approximate poly(k)-design

Unitary *k*-designs

Previous work:

(DiVincenzo, Leung, Terhal ’02) Clifford group is an *exact* 2-design

(Dankert el al ’06) Efficient construction of 2-design

(Ambainis and Emerson ’07) Efficient construction of *state*
poly(n)-design

(Harrow and Low ’08) Efficient construction of (n/log(n))-design

(Abeyesinghe ‘06) 2-designs are enough for decoupling

(Low ‘09) Other applications of *t*-design (mostly 2-designs)
replacing Haar unitaries

Random Quantum Circuits vs Unitary Designs

Previous work:

(Oliveira, Dalhsten, Plenio ’07) $O(n^3)$ random circuits are 2-designs

(Harrow, Low ’08) $O(n^2)$ random Circuits are 2-designs for every universal gate set

(Arnaud, Braun ’08) numerical evidence that $O(n\log(n))$ random circuits are unitary t-design

(Znidaric ’08) connection with spectral gap of a mean-field Hamiltonian for 2-designs

(Brown, Viola ’09) connection with spectral gap of Hamiltonian for t-designs

(B., Horodecki ’10) $O(n^2)$ local random circuits are 3-designs

Random Quantum Circuits as *k*-designs?

Conjecture Random Circuits of size poly(n, log(1/ε)) are an ε-approximate unitary poly(n)-design

Random Quantum Circuits as *k*-designs?

Thm 1 Local Random Circuits of size $O(nk^4\log(1/\varepsilon))$ are an ε-approximate unitary k-design

Thm 2 Parallel Local Random Circuits of size $O(k^4\log(1/\varepsilon))$ are an ε-approximate unitary k-design

Equilibration for Random Circuits

Thm Let RC_t := { U : U = $U_1 \ldots U_t$} be the set of all circuits of length *t*

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   for $t \geq O(\log(1/\varepsilon)\mathrm{size}(S))$

   with $\rho_S(U) := \mathrm{tr}_{\setminus S}(U|0\rangle\langle 0|^{\otimes n} U^\dagger)$

Proof follows from the calculation for a 2-design from before

Equilibration for Random Circuits

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2. Let M_k := { M : 0 ≤ M ≤ id, M has gate complexity k }.

For every $t \geq O(k^6 \log(1/\varepsilon))$

$$\mathbb{E}_{U \in RC_t} \max_{N \in M_k} \left| \langle 0^n | U^\dagger N U | 0^n \rangle - \mathrm{tr}(N)/2^n \right| \leq \varepsilon$$

Proof follows by looking at higher moments to obtain a good concentration bound on $\langle 0^n | U^T N U | 0^n \rangle$, for *fixed* N, and take the union bound over the set M_k.

Requires approximate poly(k)-design

Outline Proof of Main Result

1. Mapping the problem to bounding spectral gap of a Local Hamiltonian

2. Technique for bounding spectral gap (Nachtergaele ‘94) + representation theory
   (reduces the problem to obtaining an exponentially small lower bound on the spectral gap)

3. Path Coupling applied to the unitary group
   (prove convergence of the random walk in exponential time)

4. Use detectability Lemma (Arad et al ‘10) to go from local random circuits to *parallel* local random circuits

Relating to Spectral Gap

μ_n : measure on $U(2^n)$ induced by one step of the local random circuit model

$(\mu_n)^{*k}$: *k*-fold convolution of μ_n (measure induced by k steps of the local random circuit model)

By eigendecomposition

$$\int \mu_n(dU)U^{\otimes t,t} = \int \mu_{\text{Haar}}(dU)U^{\otimes t,t} + \sum_l \lambda_l P_l, \quad |\lambda_l| < 1$$

so $$\left\| \int \mu_n^{*t}(dU)U^{\otimes t,t} - \int \mu_{\text{Haar}}(dU)U^{\otimes t,t} \right\|_\infty = \lambda_2^t$$

Relating to Spectral Gap

μ_n : measure o

circuit mo

$(\mu_n)^{*k}$: *k*-fold (e

local ra

It suffices to a prove upper bound on λ_2 of the form $1 - \Omega(t^{-4}n^{-1})$ since

$(1 - \Omega(t^{-4}n^{-2}))^k \leq \varepsilon$ for $k = O(nt^4\log(1/\varepsilon))$

By eigendecomposition

$$\int \mu_n(dU)U^{\otimes t,t} = \int \mu_{\text{Haar}}(dU)U^{\otimes t,t} + \sum_l \lambda_l P_l, \quad |\lambda_l| < 1$$

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Relating to Spectral Gap

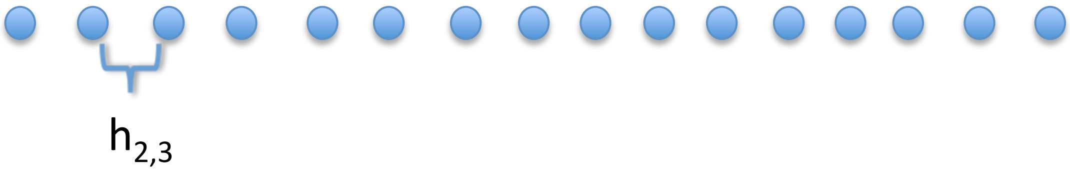
But $\mu_n = \frac{1}{n}\sum_{i=1}^{n}\mu_{Haar}(i,i+1)$

So $\lambda_2\left(\int \mu_n(dU)U^{\otimes t}\otimes \bar{U}^{\otimes t}\right) = 1 - \frac{\Delta(H_{n,t})}{n}$

with $H_{n,t} := \sum_{i=1}^{n} h_{i,i+1}$ $\quad h_{i,i+1} := I - \int_{U(4)} U_{i,i+1}^{\otimes t}\otimes \bar{U}_{i,i+1}^{\otimes t}\mu_H(dU)$

and $\Delta(H_{n,t})$ the spectral gap of the local Hamiltonian $H_{n,t}$

$H_{n,t}$:



Relating to Spectral Gap

Want to lower bound spectral gap by $O(t^{-4})$

But $\mu_n = \frac{1}{n}\sum_{i=1}^{n}\mu_{Haar}(i,i+1)$

So $\lambda_2\left(\int \mu_n(dU)U^{\otimes t}\otimes \bar{U}^{\otimes t}\right) = 1 - \frac{\Delta(H_{n,t})}{n}$

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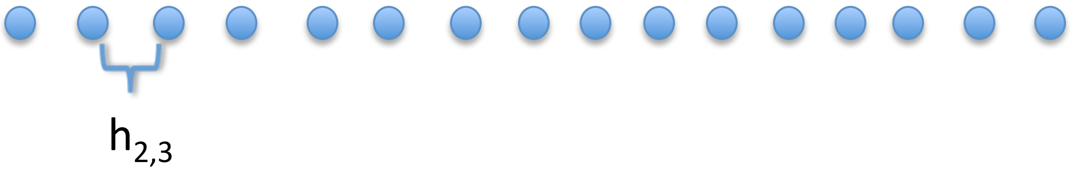


Structure of $H_{n,t}$

$$H_{n,t} := \sum_{i=1}^{n} h_{i,i+1} \qquad h_{i,i+1} := I - \int_{U(4)} U_{i,i+1}^{\otimes t} \otimes \bar{U}_{i,i+1}^{\otimes t} \mu_H(dU)$$

i. $H_{n,t} \in \left(\left(\mathbb{C}^2\right)^{\otimes 2t}\right)^{\otimes n}$, frustration-free with min eigenvalue 0

ii. $\int \mu_{\mathrm{Haar}}(dU) U^{\otimes t} \otimes \overline{U}^{\otimes t}$ projects onto 0 eigenspace $G_{n,t}$:

$$G_{n,t} := span\left\{ \left|\psi_\pi\right\rangle^{\otimes n}, \left|\psi_\pi\right\rangle := (I \otimes V(\pi)) \left|\Phi(2^t)\right\rangle : \pi \in S_t \right\}$$

$$V(\pi)|a_1, \ldots, a_t\rangle = |a_{\pi^{-1}(1)}, \ldots, a_{\pi^{-1}(t)}\rangle$$

$$|\Phi(d)\rangle = d^{-1/2} \sum_{i=1}^{d} |i, i, \rangle$$

Approximate Orthogonality

$|\psi_\pi\rangle$ are non-orthogonal, but

$$\sum_{\pi\in S_t} \left|\langle\psi_\sigma|\psi_\pi\rangle\right|^n \leq 1+\frac{2t^2}{2^n}, \qquad \left\|\sum_{\pi\in S_t}\left(|\psi_\pi\rangle\langle\psi_\pi|\right)^{\otimes n} - G_{n,t}\right\|_\infty \leq \frac{2t^2}{d^n}$$

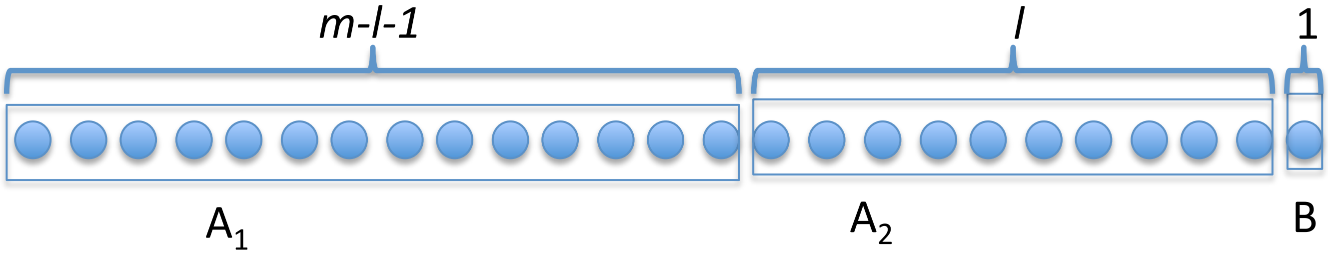
Proof by basic representation theory, in fact only uses

$$P_{sym} = \frac{1}{t!}\sum_{t\in S_t} V(\pi)$$

Lower Bounding $\Delta(H_{n,t})$

Lemma: $\Delta(H_{n,t}) \geq \dfrac{\Delta(H_{\log(t),t})}{8\log(t)}$

Follows from structure of $H_{n,t}$, approx. orthogonality, *and*



(Nachtergaele '96) Suppose there exists l and $\varepsilon_l < l^{-1/2}$ s.t.

$$\left\| (I_{A_1} \otimes G_{A_2B})(G_{A_1A_2} \otimes I_B) - G_{A_1A_2B} \right\|_\infty \leq \varepsilon_l$$

Then: $\Delta(H_{[1,n]}) \geq \Delta(H_{[1,l]}) \left(\dfrac{\left(1 - \varepsilon_l \sqrt{l}\right)}{l-1} \right)$

Lower Bounding $\Delta(H_{n,t})$

Lemma: $\Delta(H_{n,t}) \geq \frac{\Delta(H_{\log(t),t})}{8\log(t)}$

Follo

Want to lower bound by O(t^{-4}), an exponential small bound in the size of the chain (i.e. in log(t))

(Nachtergaele '96) Suppose there exists *l* and $\varepsilon_l < l^{-1/2}$ s.t.

$$\left\|(I_{A_1}\otimes G_{A_2B})(G_{A_1A_2}\otimes I_B)-G_{A_1A_2B}\right\|_\infty \leq \varepsilon_l$$

Then: $$\Delta(H_{[1,n]}) \geq \Delta(H_{[1,l]})\left(\frac{\left(1-\varepsilon_l\sqrt{l}\right)}{l-1}\right)$$

Exponentially Small Bound to Spectral Gap

Follows from two relations:

1. $$\left(1-\frac{\Delta(H_{n,t})}{n}\right)^k \leq 2tW\left((\mu_n)^{*k}, \mu_{Haar}\right)$$

Wasserstein distance:

$$W(\nu_1,\nu_2) := \sup\left\{\int f(u)\nu_1(du) - \int f(u)\nu_2(du) : f \text{ is } 1-Lipschitz\right\}$$

2. $$W\left((\mu_n)^{*(n-1)k}, \mu_{Haar}\right) \leq 2^{n/2}(1-2^{-5n})^{\frac{k}{n-1}}$$

Bounding Convergence with Path Coupling

Key result to 2nd relation: Extension to the unitary group of Bubley and Dyer path coupling

Let $W_p\left(\nu_1,\nu_2\right) := \inf\left\{E[d(X,Y)^p]^{1/p} : (X,Y) \ couples \ (\nu_1,\nu_2)\right\}$

(Oliveira '07) Let ν be a measure in U(d) s.t.

$$\lim_{\varepsilon\to 0} \sup_{U_1,U_2\in U(d)} \left\{ \frac{W_2\left(\nu * \delta_{U_1}, \nu * \delta_{U_2}\right)}{\left\|U_1 - U_2\right\|_2} : \left\|U_1 - U_2\right\|_2 \leq \varepsilon \right\} \leq \eta$$

Then $$W_2(\nu * \nu_1, \nu * \nu_2) \leq \eta W_2(\nu_1, \nu_2)$$

Bounding Convergence with Path Coupling

Must consider coupling in n steps of the walk to get non trivial contraction (see paper for details)

(Oliveira ‘07) Let ν be a measure in U(d) s.t.

$$\lim_{\varepsilon \to 0} \sup_{U_1, U_2 \in U(d)} \left\{ \frac{W_2\left(\nu * \delta_{U_1}, \nu * \delta_{U_2}\right)}{\left\| U_1 - U_2 \right\|_2} : \left\| U_1 - U_2 \right\|_2 \leq \varepsilon \right\} \leq \eta$$

Then $$W_2(\nu * \nu_1, \nu * \nu_2) \leq \eta W_2(\nu_1, \nu_2)$$

Time-Independent Models

Toy model for equilibration: Let $H_{SE} = UDU^T$, with U taken from the Haar measure in U(|S||E|) and D := diag(E_1, E_2, ….).

$$\frac{1}{T}\int_{U(|S||E|)} tr\left(\left(\rho_S(t)-\rho_{equi}\right)^2\right)\mu_{Haar}(dU) \leq O\left(\frac{\left(\sum_k e^{i2tE_k}\right)^2}{|S||E|^2}+\ldots\right)$$

(B., Ciwiklinski et al '11, Masanes et al '11, Vinayak, Znidaric '11)

Time of equilibration: Average energy gap: $\frac{1}{|S|^2|E|^2}\sum_{j,l}(E_j - E_l)^{-1}$

For typical eigenvalue distribution goes with O(1/log(|E|))

Fast Equilibration

Calculation only involves 4th moments:

$$\frac{1}{T}\int_{U(|S||E|)} tr\left(\left(tr_E\left(Ue^{-itD}U'\rho_0 Ue^{itD}U'\right)-\rho_{equi}\right)^2\right)\mu_{Haar}(dU)$$

Can replace Haar measure by an approximate unitary 4-design

Cor For most Hamiltonians of the form UDU^T with *U* a random quantum circuit of O(n^2) size, small subsystems equilibrate fast.

Open Questions

- What happens in higher dimensions?

- Fast scrambling conjecture (Hayden et al ‘11)
  Do O(log(n))-depth random circuits equilibrate?
  ((Brown, Fawzi ’13) true for depth O(log^2(n)))

- Equilibration for time-independent *local* Hamiltonians?
  ((B., Ciwiklinski et al ‘11, Masanes et al ‘11, Vinayak, Znidaric ’11) time-independent non-local Ham.)

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Thanks!