Entanglement Spectrum, Topological Entanglement Entropy and a Quantum Hammersley-Clifford Theorem

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based on joint work with

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MIT 2016

Quantum Information Theory

Goal: Lay down the theory for future quantum-based technology (quantum computers, quantum cryptography, …)

Quant. Comm.
Entanglement theory
Q. error correc. + FT
Quantum comp.
Quantum complex. theo.

Quantum Information Theory

Goal: Lay down the theory for future quantum-based technology (quantum computers, quantum cryptography, …)

Ultimate limits to information transmission

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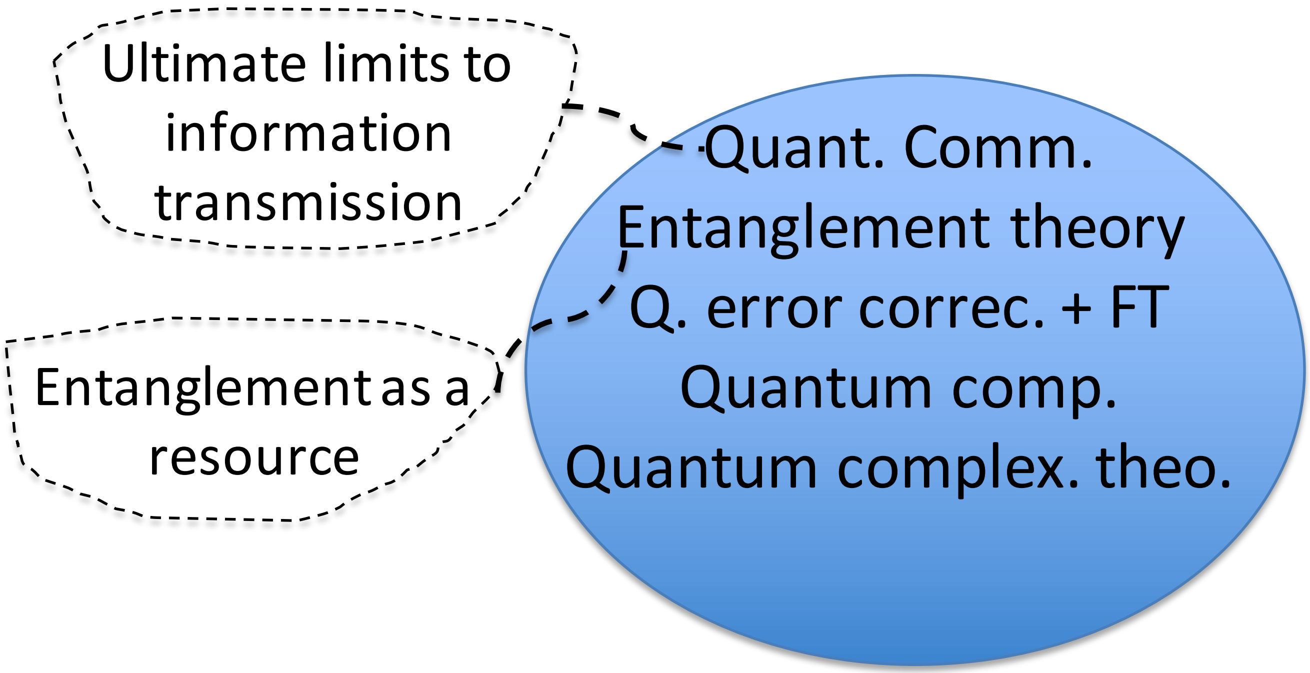

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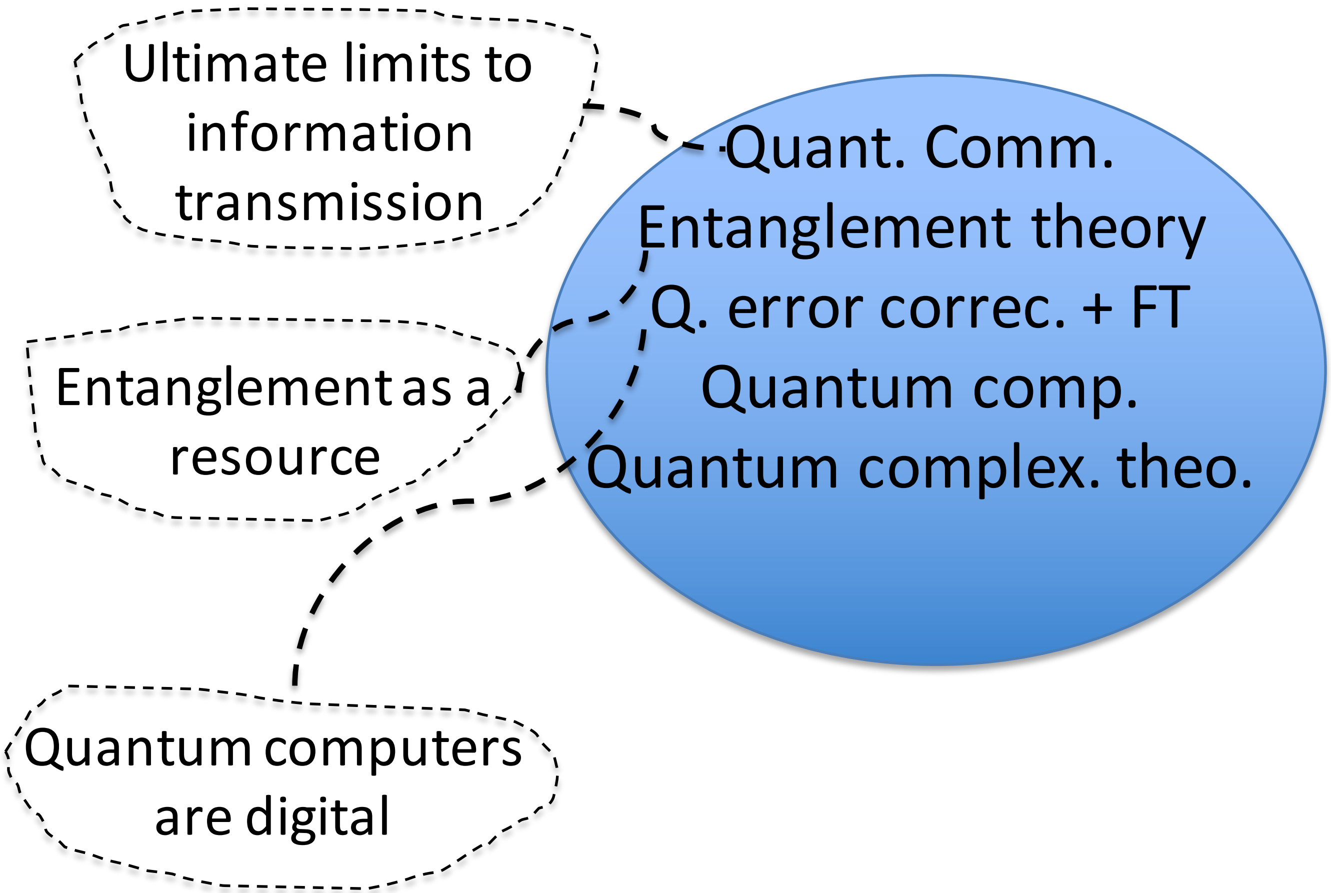

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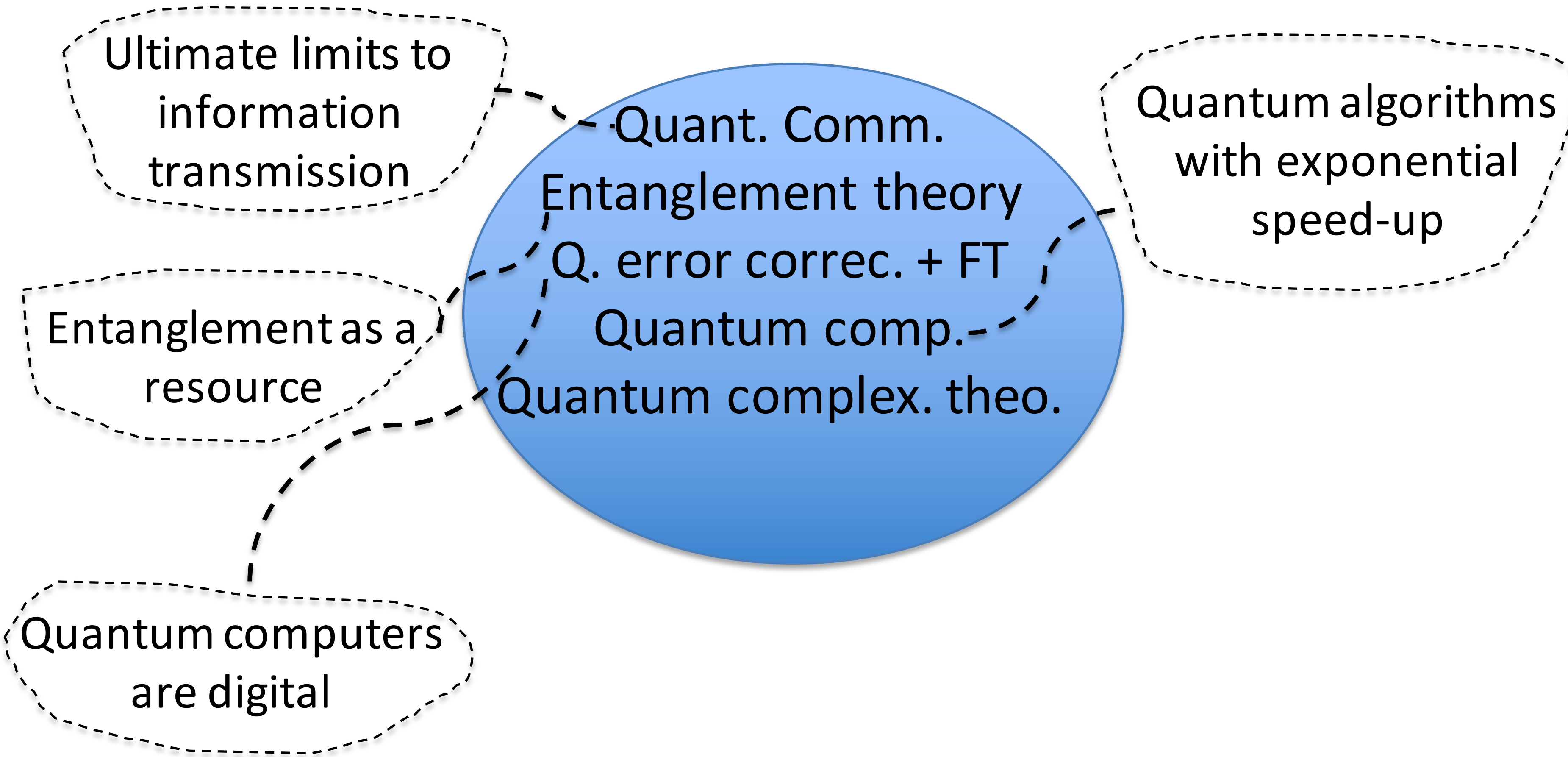

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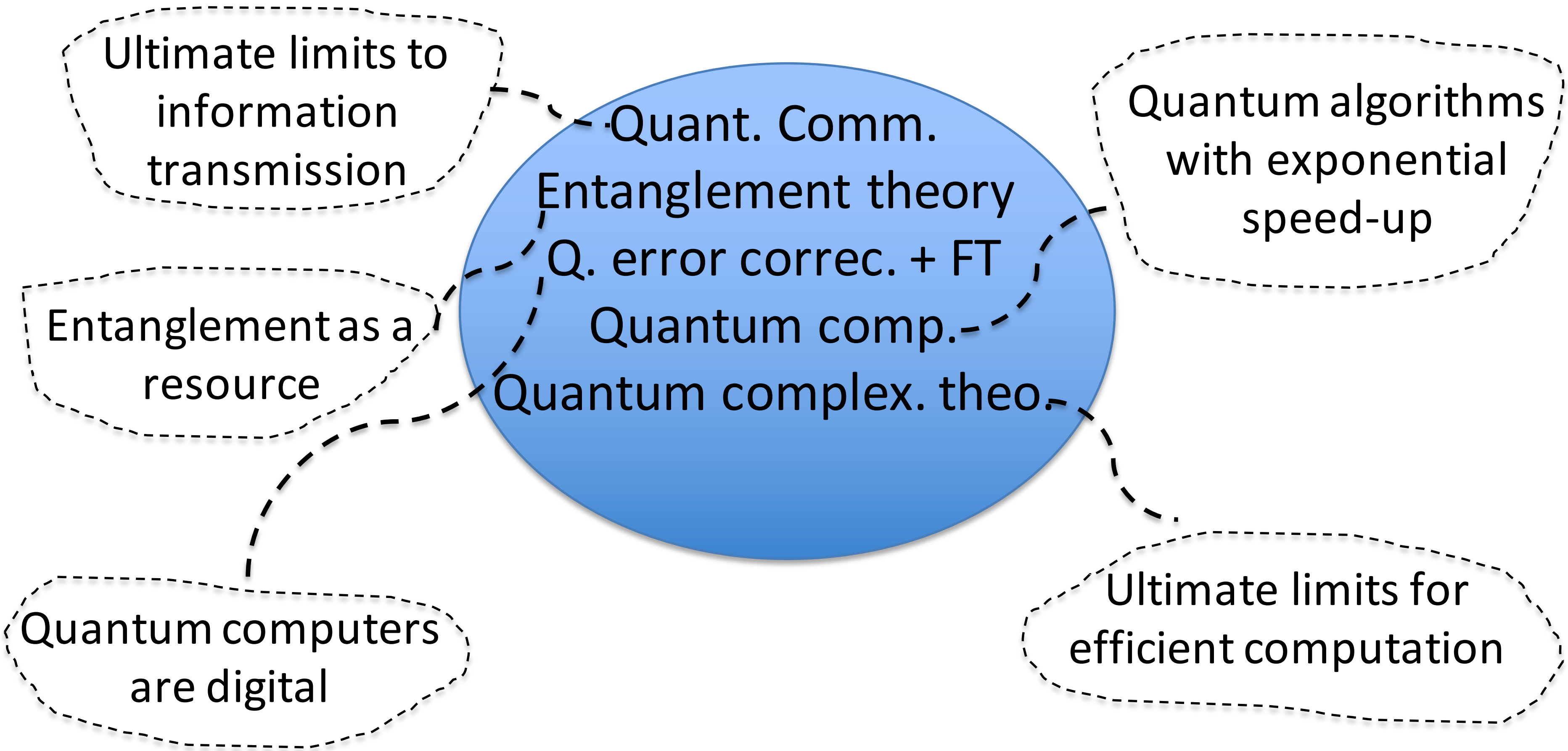
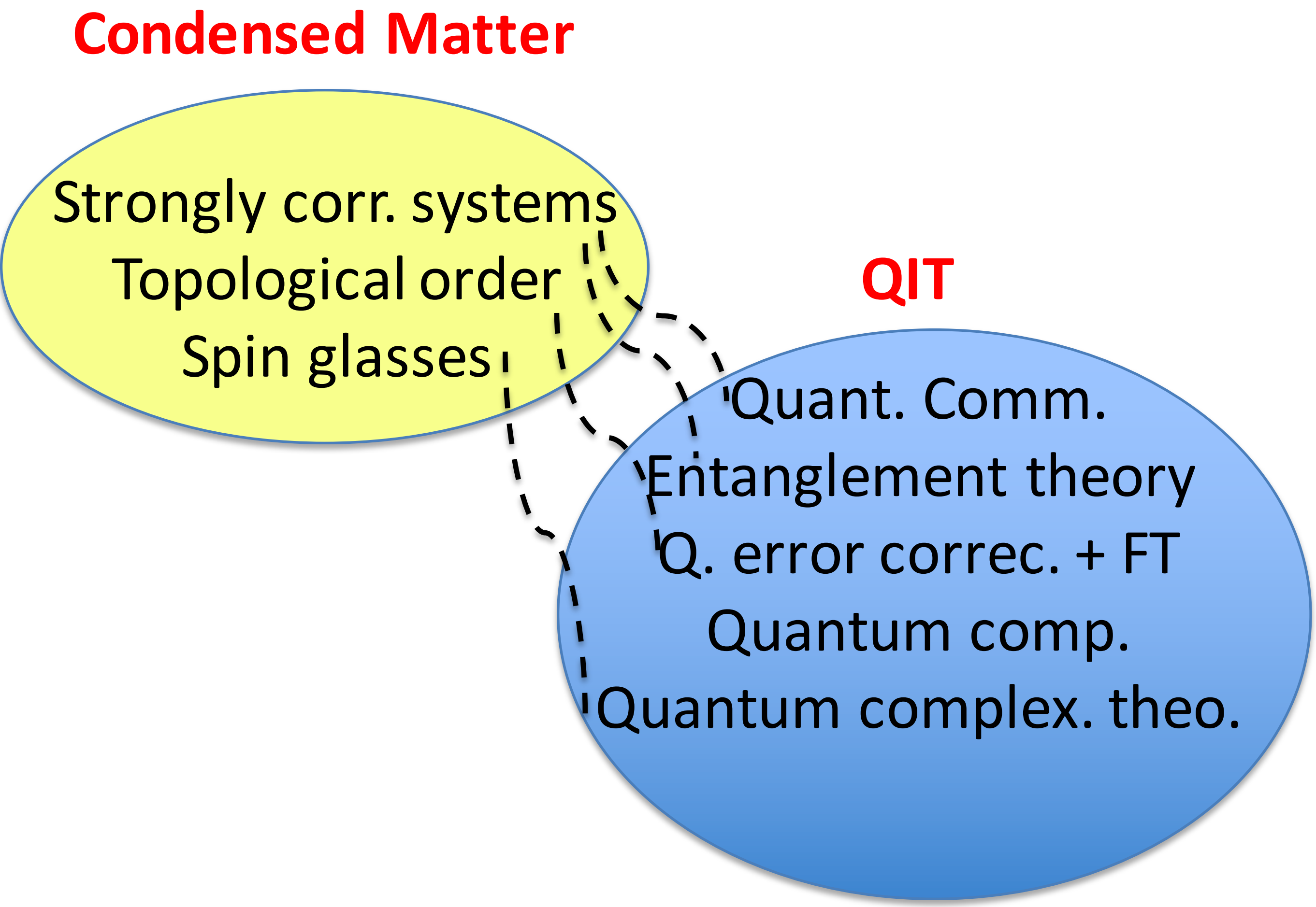


QIT Connections

QIT

- Quant. Comm.
- Entanglement theory
- Q. error correc. + FT
- Quantum comp.
- Quantum complex. theo.

QIT Connections



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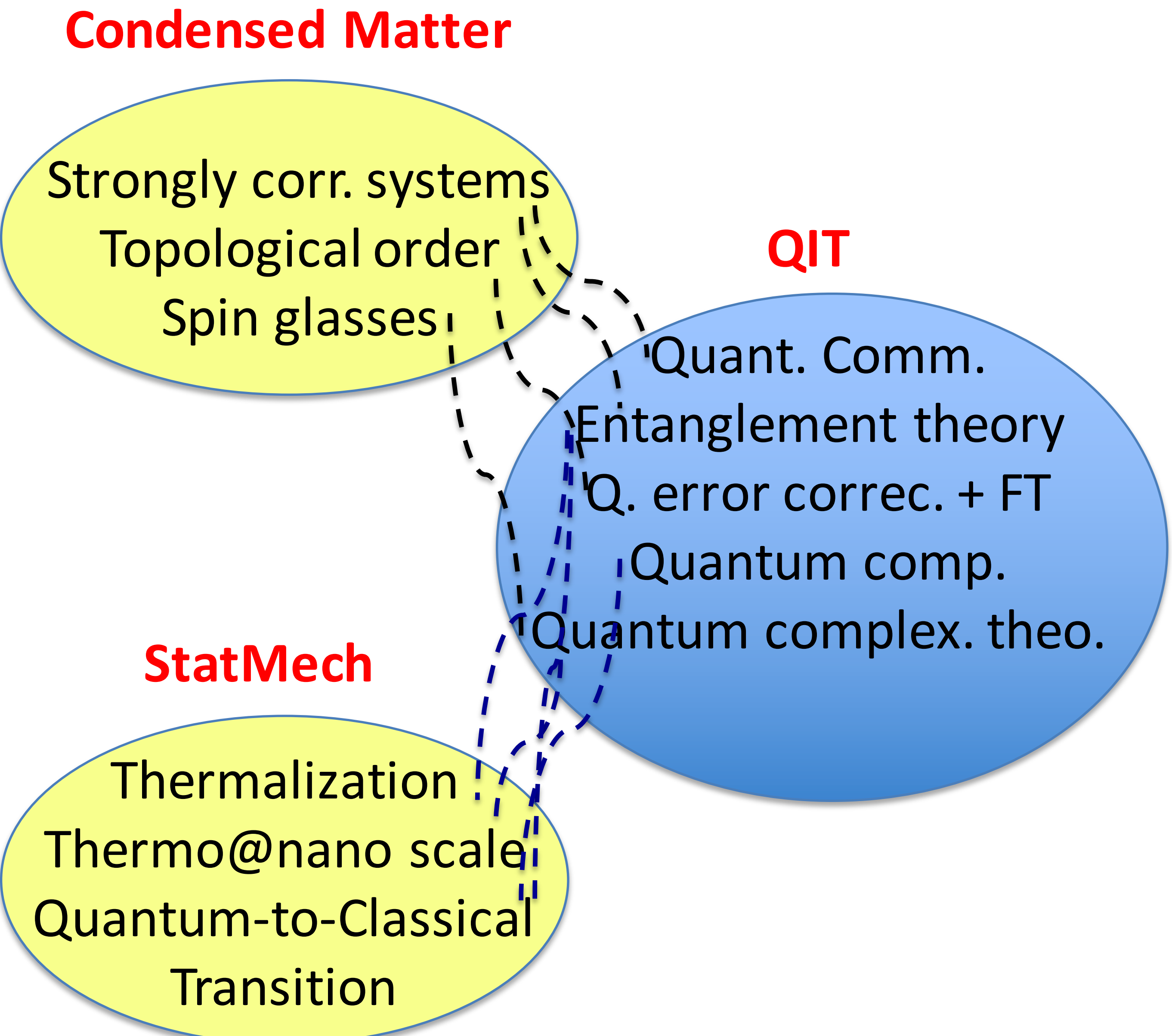

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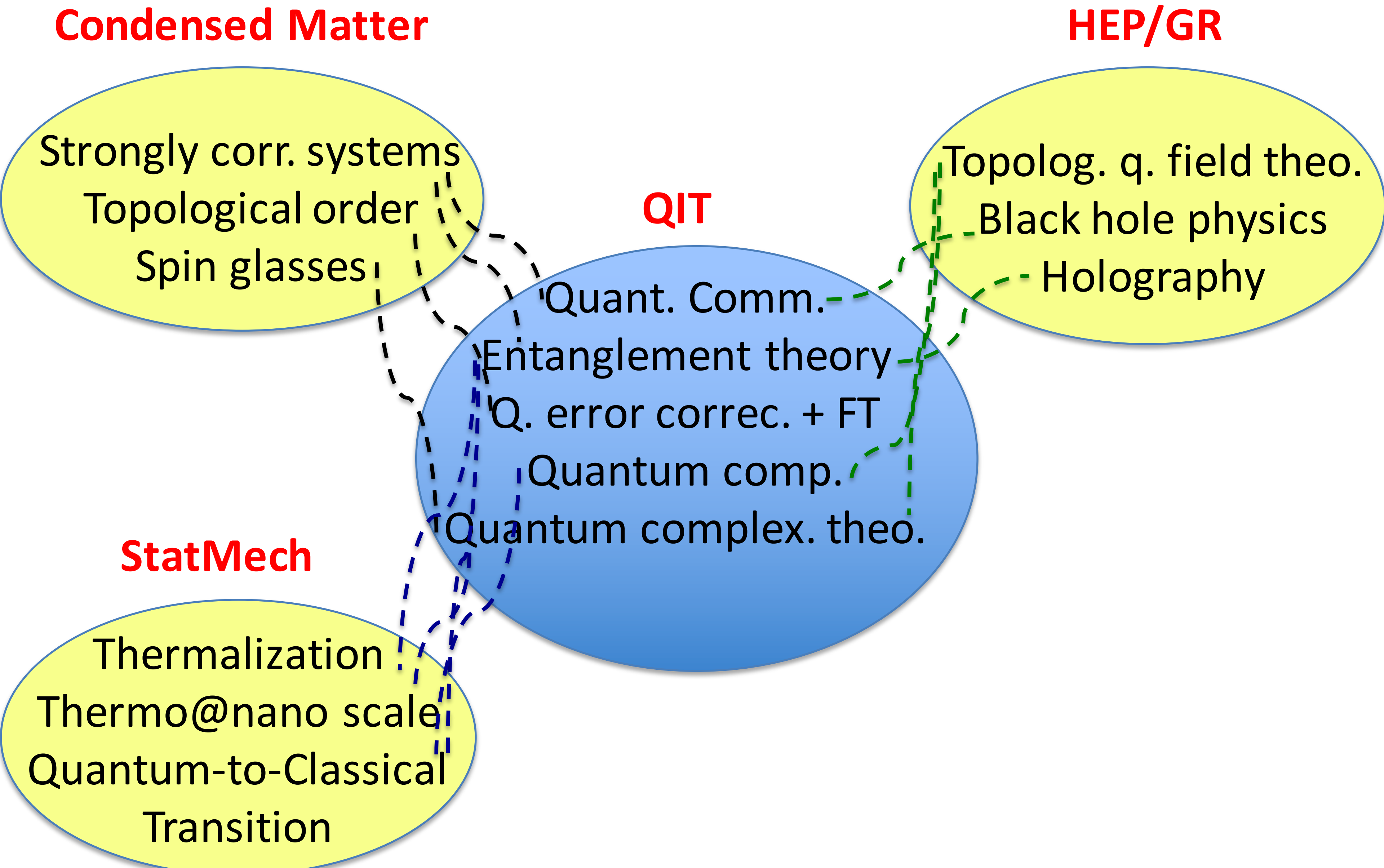

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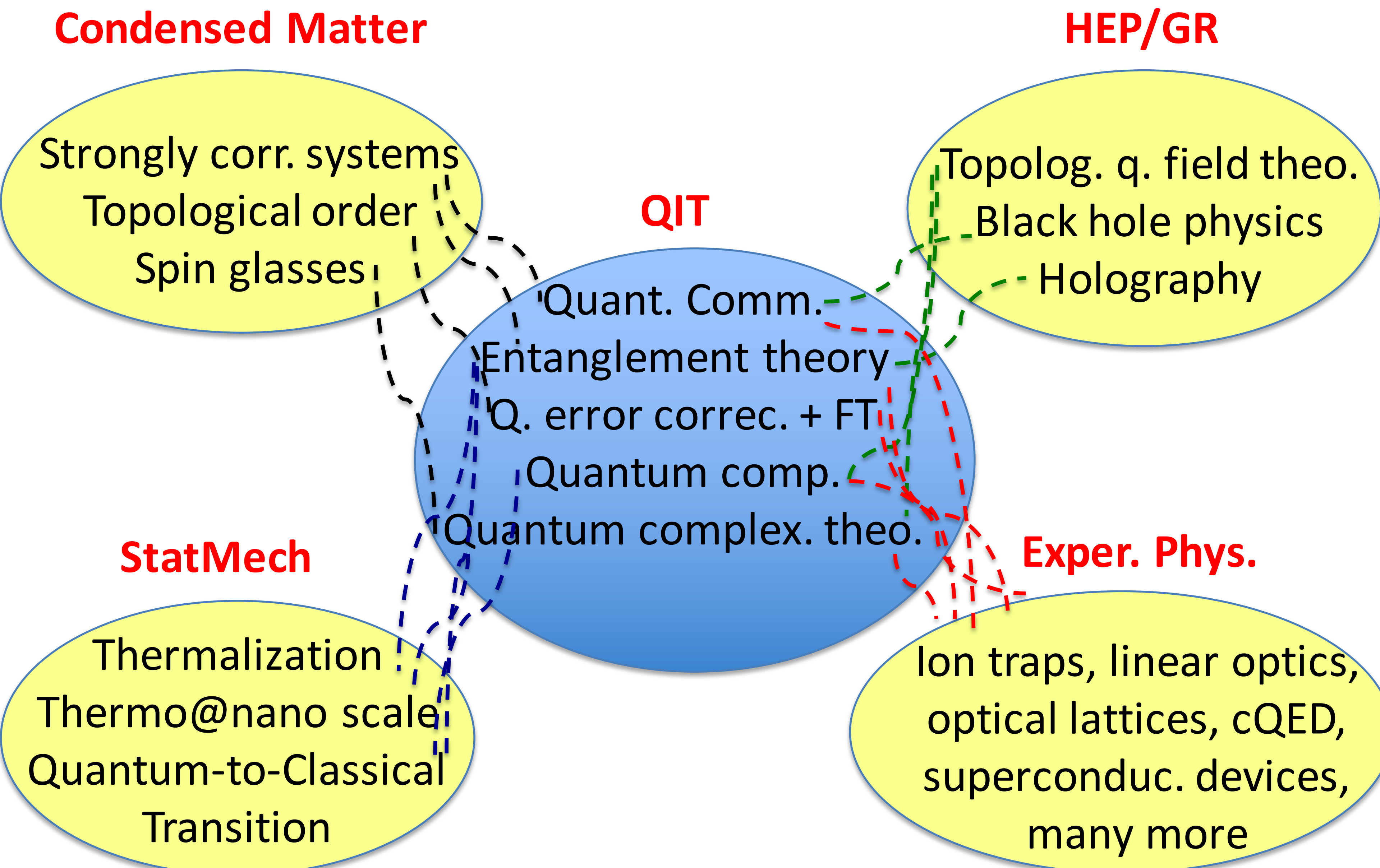
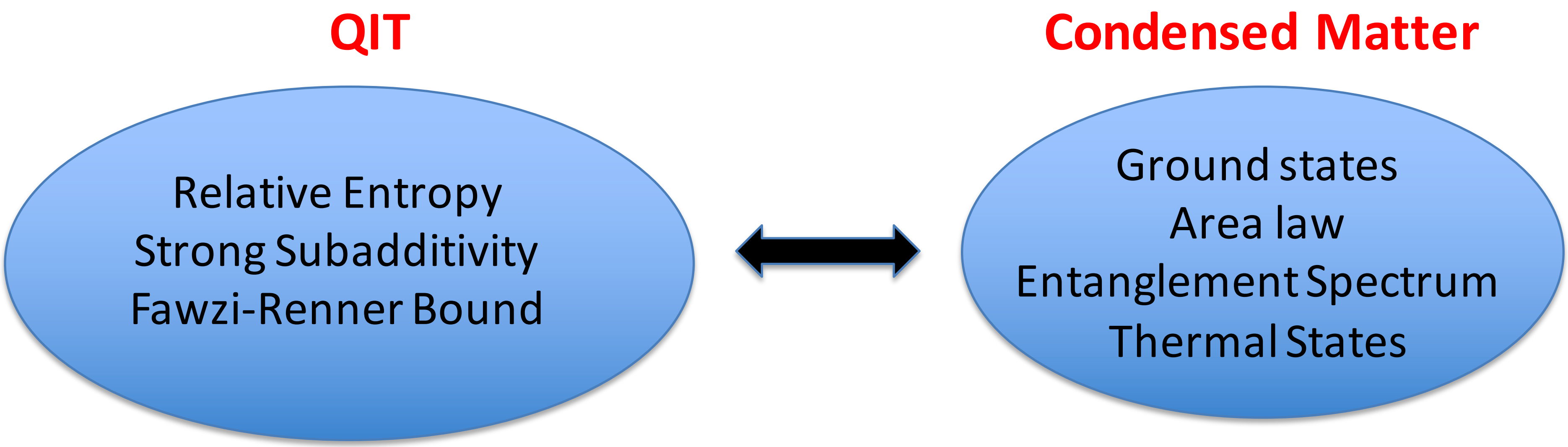


This Talk

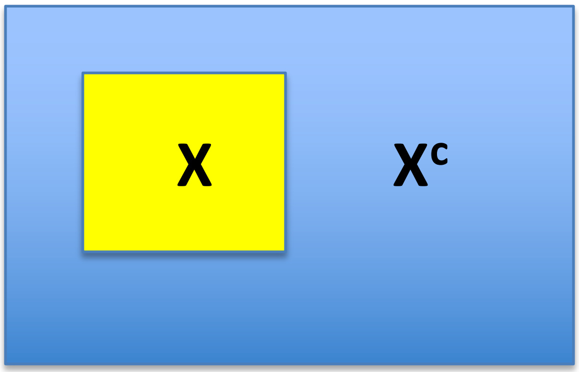
Goal: give one example of these emerging connections:

Study scaling of *entanglement* in some physical states



Area Law

$|\psi\rangle_{XX^c}$



Area law assumption: For every region *X*,

$$S(X) = a|\partial X| - \gamma + \exp(-c|\partial|/\xi)$$

Topological entanglement entropy
(Kitaev, Preskill '05, Levin, Wen '05)

correlation length

Expected to hold in models with a correlation length.
But not always true.

In this talk we consider this form of area law as an assumption and analyse what are its consequences.

Quantum Information 1.01: Fidelity

… it's a measure of distinguishability between two quantum states.

Given two quantum states their fidelity is given by

$$F(\rho, \sigma) := \mathrm{tr}((\rho^{1/2}\sigma\rho^{1/2})^{1/2})$$

It tells how distinguishable they are by any quantum measurement

Quantum Information 1.01: Relative Entropy

… it's another measure of distinguishability between two quantum states.

Def: $S(\rho\|\sigma) := \mathrm{tr}(\rho(\log(\rho) - \log(\sigma)))$

Gives optimal exponent for distinguishing the two states (in asymmetric hypothesis testing; Stein's Lemma)

Pinsker's inequality: $S(\rho\|\sigma) \geq -\frac{1}{2} \log F(\rho, \sigma)$

Conditional Mutual Information

Given ρ_{ABC} ,

$$I(A:C|B) := S(AB) + S(BC) - S(ABC) - S(B)$$
$$= S(\rho_{ABC} \| \exp(\log(\rho_{AB}) + \log(\rho_{BC}) - \log(\rho_B)))$$

Strong sub-additivity: $I(A:C|B) \geq 0$

Conditional Mutual Information

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Strong sub-additivity: $I(A:C|B) \geq 0$

Fawzi-Renner '14:

$$I(A:C|B) \geq \frac{1}{2} \min_{\Lambda : B \to BC} - \log(F(\rho_{ABC}, \Lambda(\rho_{AB})))$$

Conditional Mutual Information

Given ρ_{ABC},

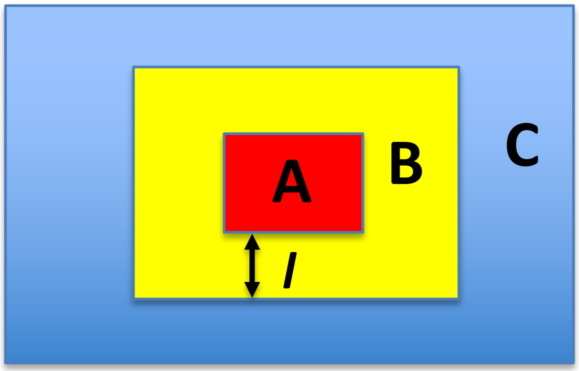
$$
\begin{aligned}
I(A:C|B) &:= S(AB) + S(BC) - S(ABC) - S(B) \\
&= S(\rho_{ABC} \| \exp(\log(\rho_{AB}) + \log(\rho_{BC}) - \log(\rho_B)))
\end{aligned}
$$

Strong sub-additivity: $I(A:C|B) \geq 0$

(Fawzi-Renner ‘14) If $I(A:C|B) \leq \varepsilon$, there is a channel $\Lambda : B \to BC$ s.t. $F(\rho_{ABC}, \Lambda^{B \to BC}(\rho_{AB})) \geq 1 - 2\varepsilon$

Can reconstruct the state ABC from reduction on AB by acting on B only

Consequence of Area Law: State Reconstruction



Area law assumption: For every region *X*,

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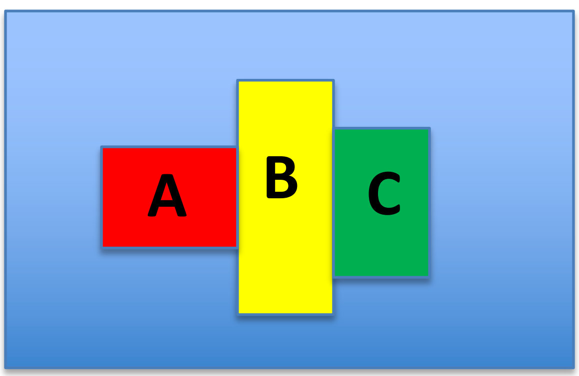
Topological entanglement entropy

correlation length



For every ABC with trivial topology:

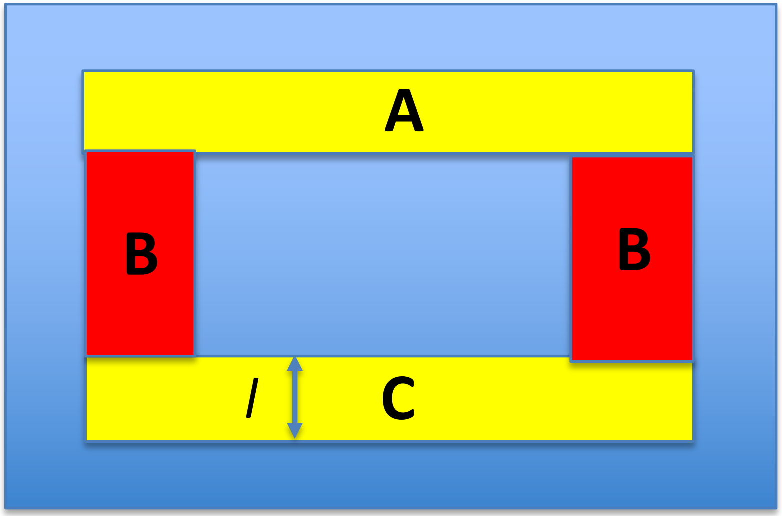
$$I(A:C|B) \leq \exp(-cl)$$

(Kitaev '12) $\gamma = 0$ implies the state can be created by short-depth circuit

(Kim '14) Implies the state can be constructed from local parts

Topological Entanglement Entropy

(Kitaev, Preskill '05, Levin, Wen '05)



Area law assumption: For every region *X*,

$$S(X) = a|\partial X| - \gamma + \exp(-c|\partial|/\xi)$$

Topological entanglement entropy (γ)

correlation length (ξ)

Conditional Mutual Information:

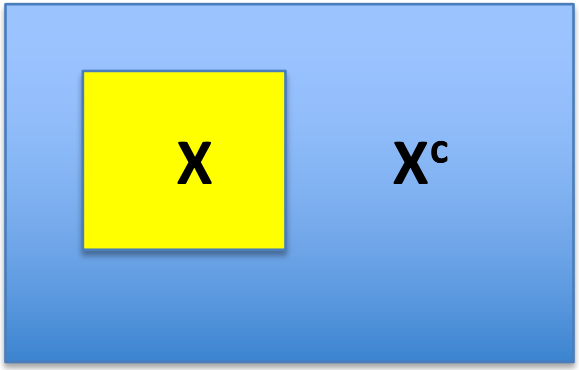
$$I(A:C|B) := S(AB) + S(BC) - S(ABC) - S(B)$$

Assuming area law holds:

$$I(A:C|B) = 2\gamma + \exp(-c'l)$$

Entanglement Spectrum

$|\psi\rangle_{XX^c}$



$\lambda(\rho_X)$: eigenvalues of reduced density matrix on X

Also known as Schmidt eigenvalues of the state $|\psi\rangle_{XX^c}$

Entanglement Spectrum

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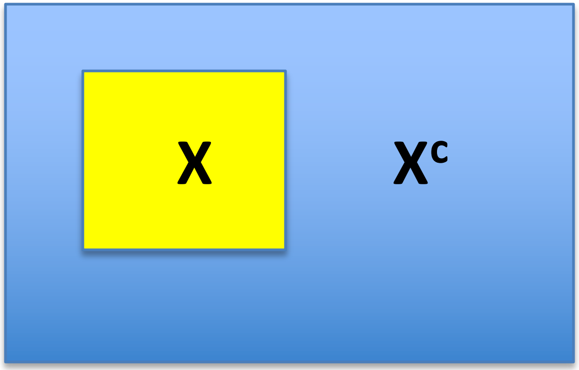

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(Haldane, Li '08, ….)

For FQHE, entanglement spectrum matches the low energies of a CFT acting on the boundary of *X*

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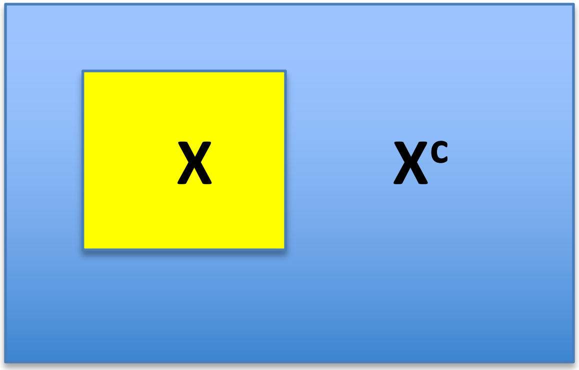

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For FQHE, entanglement spectrum matches the low energies of a CFT acting on the boundary of *X*

(Cirac, Poiblanc, Schuch, Verstraete '11, ….)

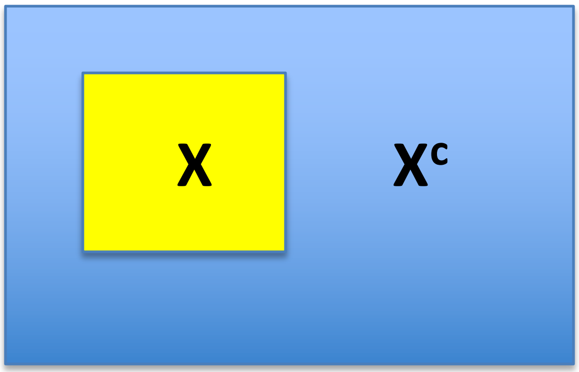
Numerical studies with PEPS.

For *topologically trivial* systems (AKLT, Heisenberg models): entanglement spectrum matches the energies of a *local Hamiltonian* on boundary

For *topological* systems (Toric code): needs *non-local* Hamiltonian

Entanglement Spectrum

$|\psi\rangle_{XX^c}$



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For *topologically trivial* systems (AKLT, Heisenberg models):
entanglement spectrum matches the energies of a *local Hamiltonian* on boundary

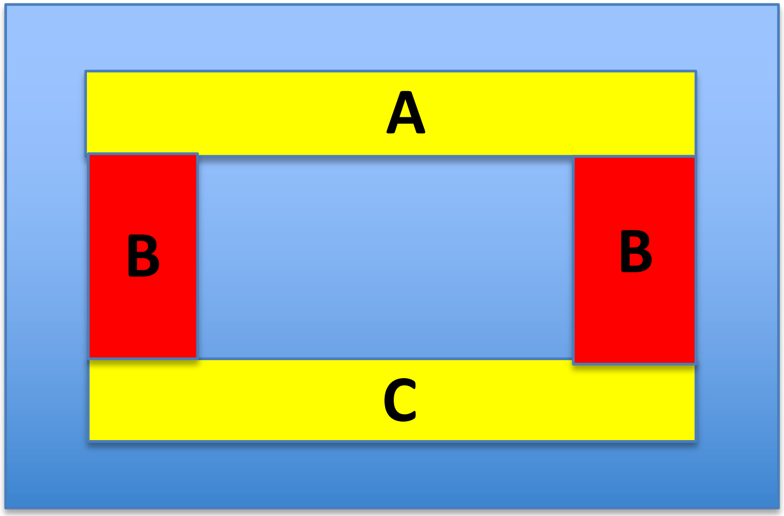
For *topological* systems (Toric code): needs *non-local* Hamiltonian

How general are these findings? Can we make them more precise?

Result 1: Boundary State

thm 1 Suppose $|\psi\rangle$ satisfies the area law assumption. Then

$$\begin{aligned} 2\gamma &\approx I(A:C|B) \\ &\approx \min_{H_{AB},H_{BC}} S(\rho_{ABC} \| \exp(H_{AB}+H_{BC})/Z) \end{aligned}$$



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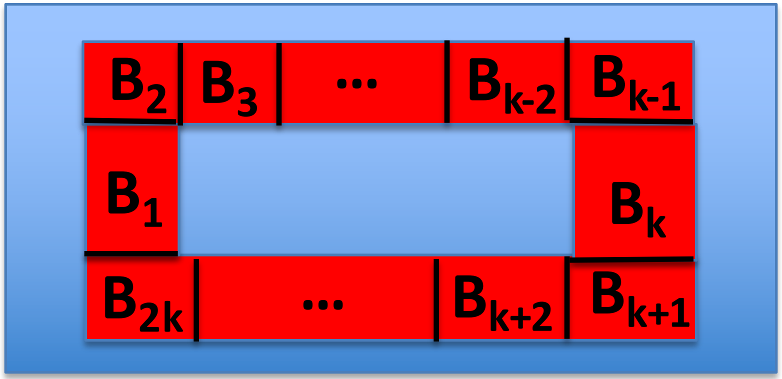
Suppose $\gamma = 0$. Then there is a local $H = \sum_k H_{B_k,B_{k+1}}$ s.t.

$$\rho_{B_1 \ldots B_l} \approx \exp(H)/Z$$



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$\gamma = 0 \implies$ Local "boundary Hamiltonian"

$\gamma \neq 0 \implies$ Non-local "boundary Hamiltonian"

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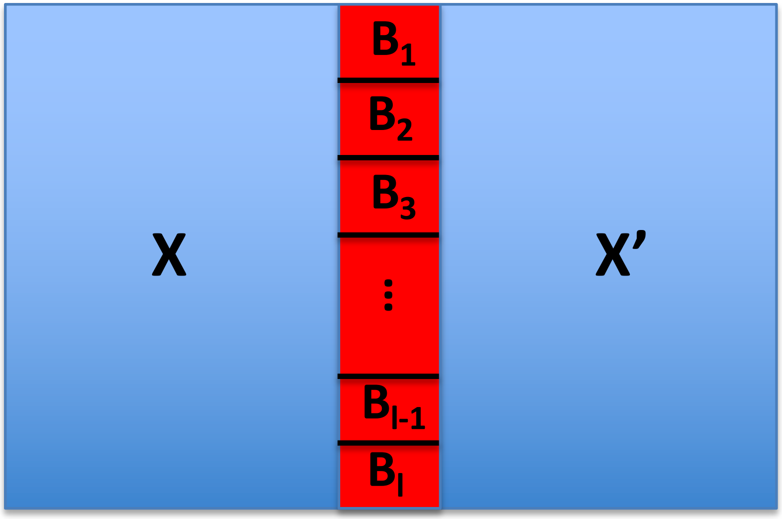
$$\rho_{B_1 \ldots B_l} \approx \exp(H)/Z$$

Obs: Correlation length of the state $|\psi\rangle$ determines temperature of the thermal state ($1/T := \max_k \|H_{B_k B_{k+1}}\|$)

Result 2: Entanglement Spectrum

thm 2 Suppose $|\psi\rangle$ satisfies the area law assumption with $\gamma = 0$. Then

$$\lambda(\rho_X)^{\otimes 2} \approx \lambda(e^{\sum_k H_{B_k, B_{k+1}}})$$



Result 2: Entanglement Spectrum

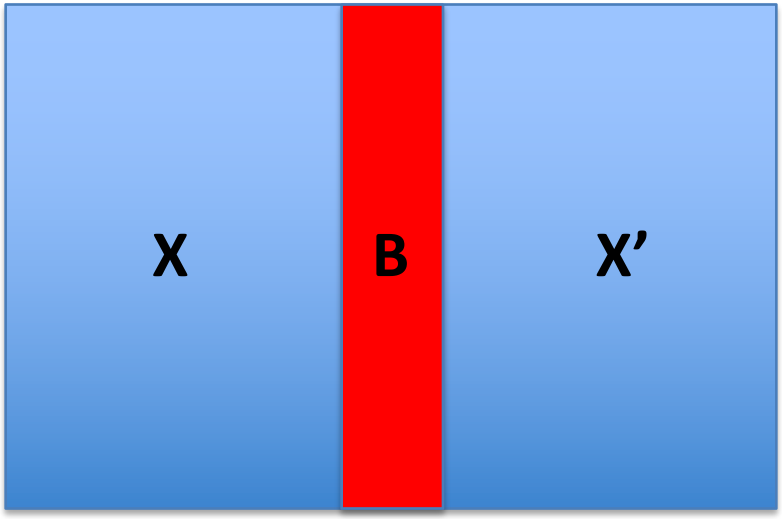
thm 2 Suppose $|\psi\rangle$ satisfies the area law assumption with $\gamma = 0$. Then

$$\lambda(\rho_X)^{\otimes 2} \approx \lambda(e^{\sum_k H_{B_k, B_{k+1}}})$$

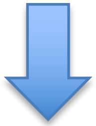
If $\gamma \neq 0$, then for every *k* there is *no* local Hamilatonian *H* s.t.

$$\lambda(\rho_X)^{\otimes k} \approx \lambda(e^{H})$$

From thm 1 to thm 2

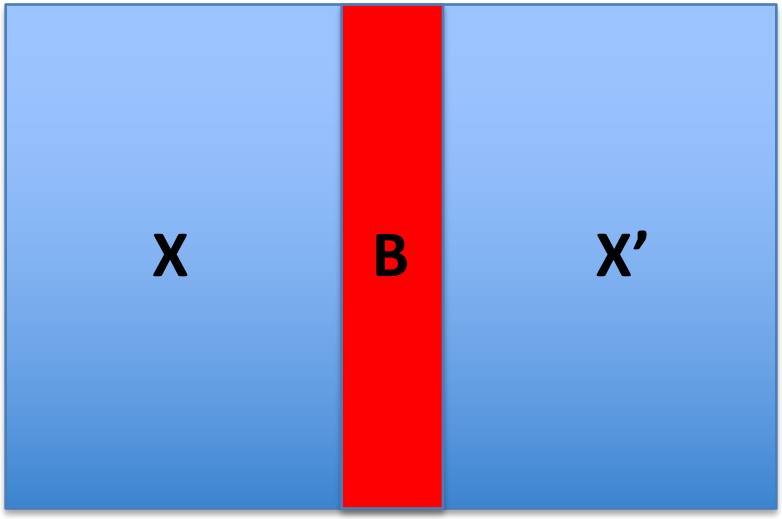


$$
\begin{aligned}
& I(X : X') \\
= \quad & I(X : X'|B) \approx 0
\end{aligned}
$$



$$\rho_{XX'} \approx \rho_X \otimes \rho_{X'}$$

From thm 1 to thm 2



$$\begin{aligned} & I(X : X') \\ = \quad & I(X : X'|B) \approx 0 \end{aligned}$$

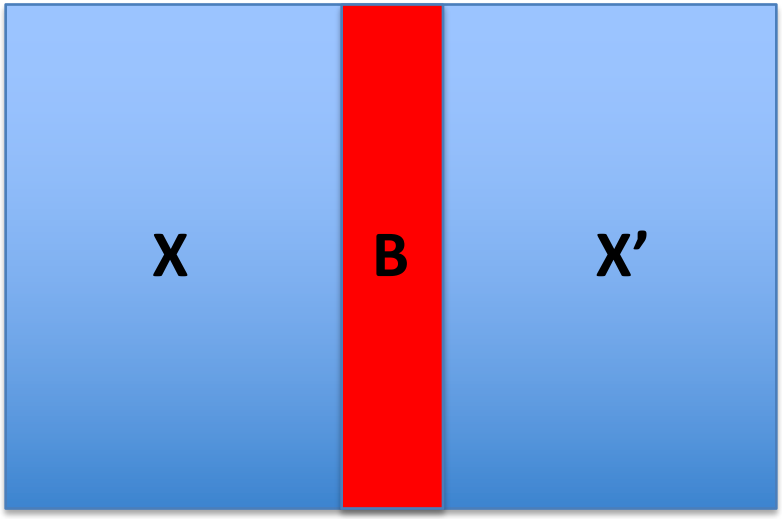
$$\Downarrow$$

$$\rho_{XX'} \approx \rho_X \otimes \rho_{X'}$$

$$\lambda(\rho_{XX'}) = \lambda(\rho_B)$$

since $|\psi\rangle_{XBX'}$ is a pure state

From thm 1 to thm 2



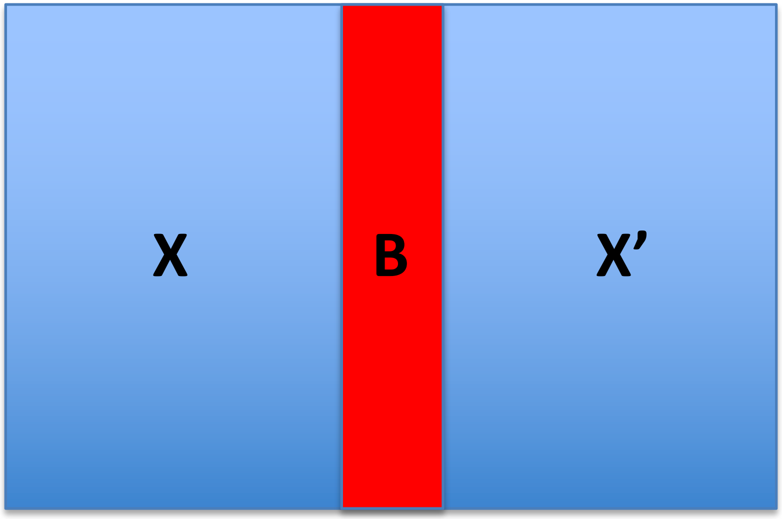
$$I(X : X') = I(X : X'|B) \approx 0$$

$$\rho_{XX'} \approx \rho_X \otimes \rho_{X'}$$

$$\lambda(\rho_{XX'}) = \lambda(\rho_B)$$

$$\Rightarrow \lambda(\rho_X) \otimes \lambda(\rho_{X'}) \approx \lambda(\rho_B)$$

From thm 1 to thm 2



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$$\rho_{XX'} \approx \rho_X \otimes \rho_{X'}$$

$$\lambda(\rho_{XX'}) = \lambda(\rho_B)$$

$$\Rightarrow \lambda(\rho_X) \otimes \lambda(\rho_{X'}) \approx \lambda(\rho_B)$$

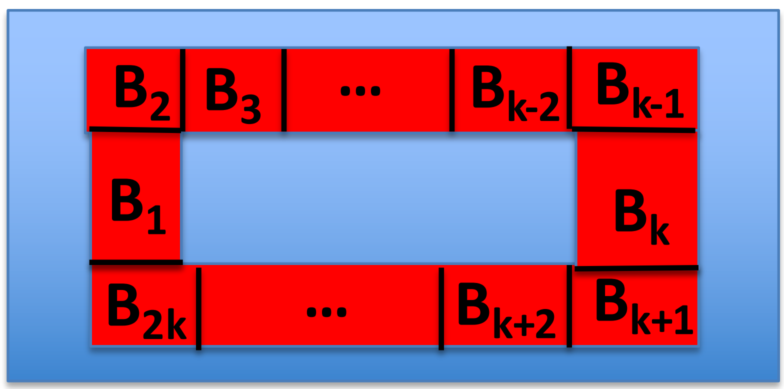
If $\gamma = 0, \quad \rho_B \approx e^{\sum_k H_{B_k, B_{k+1}}} / Z$

How to prove thm 1?

We'll start with the second part. Recap:

Suppose $\gamma = 0$. Then there is a local $H = \sum_k H_{B_k, B_{k+1}}$ s.t.

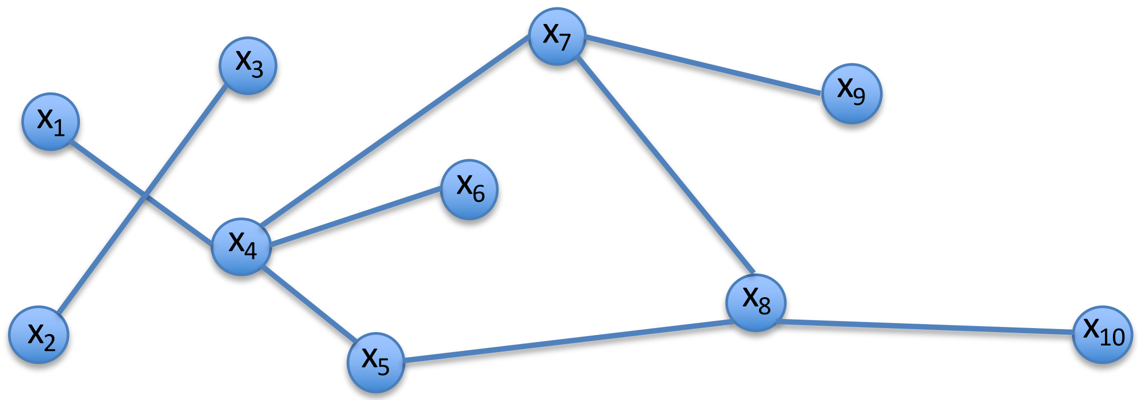
$$\rho_{B_1 \ldots B_l} \approx \exp(H)/Z$$



By area law: $I(B_1 \ldots B_i : B_{i+1} \ldots B_{2k-1} | B_i B_{2k}) \approx 0 \;\; \forall i$

The idea is to show this implies the state is approximately thermal

Markov Networks

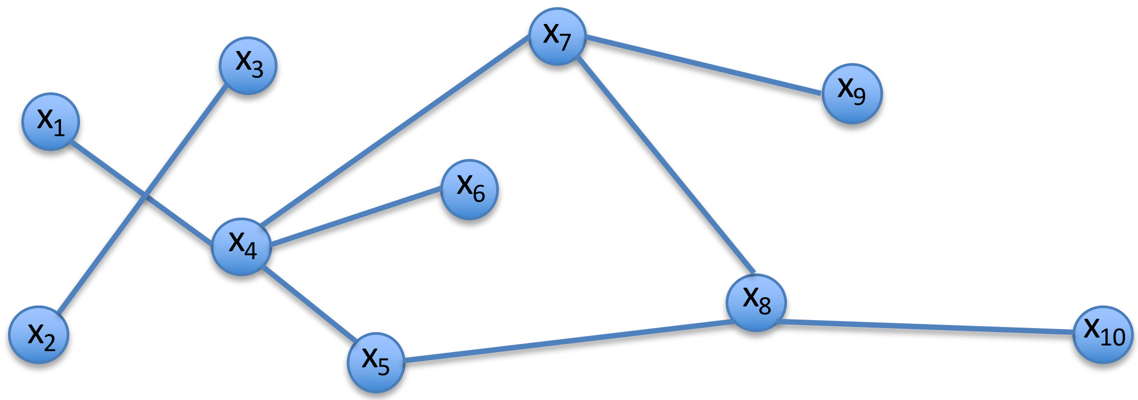


We say r.v. $x_1, \ldots, x_n$ on a graph G form a Markov Network if x_i *is* indendent of all other x's conditioned on its neighbors

I.e. Let N_i be set of neighbors of vertex i. Then for every i,

$$I(x_i : \cup_{j \neq i, j \notin N_i} x_j \,|\, \cup_{j \in N_i} x_j) = 0$$

Hammersley-Clifford Theorem

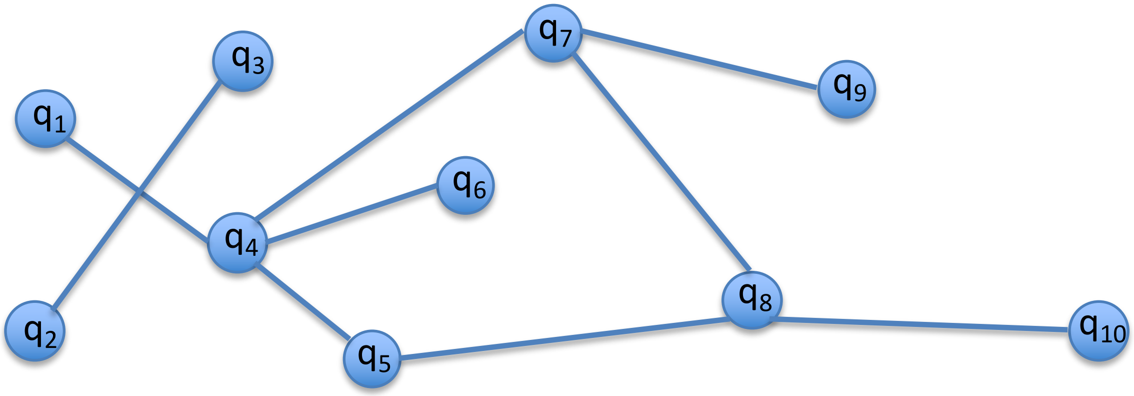


(Hammersley-Clifford '71) Let G = (V, E) be a graph and P(V) be a positive probability distribution over r.v. located at the vertices of G. The pair (P(V), G) is a Markov Network if, and only if, the probability P can be expressed as $P(V) = e^{H(V)}/Z$ where

$$H(V) = \sum_{Q \in \mathcal{C}} h_Q(Q)$$

is a sum of real functions $h_Q(Q)$ of the r.v. in cliques Q.

Quantum Hammersley-Clifford Theorem

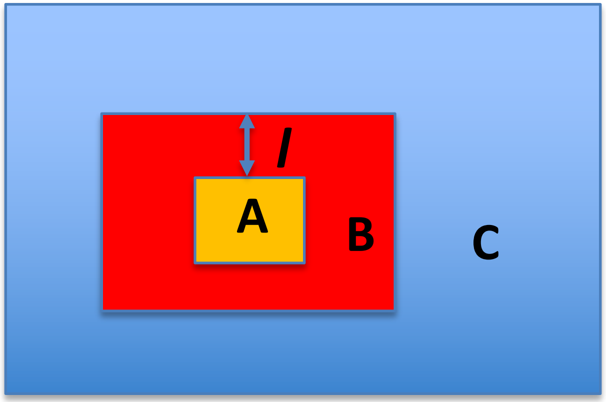


(Leifer, Poulin '08, Brown, Poulin '12) An analogous result holds replacing classical Hamiltonians by *commuting* quantum Hamiltonians

(obs: quantum version more fragile; only works for graphs with no 3-cliques)

Can we get a similar characterization for general quantum thermal states?

Approximate Quantum Hammersley-Clifford Theorem?



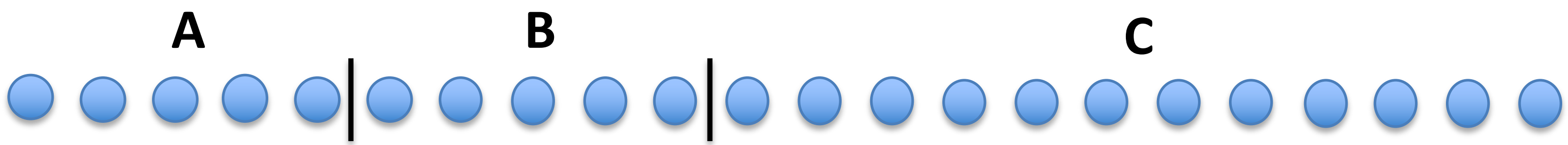
Def: We say a quantum state is a (l, eps)-approximate Markov network if for every regions ABC s.t. B shields A from C and B has width l,

$$I(A : C|B) \leq \varepsilon$$

Conjecture: Approximate Markov Networks are equivalent to Gibbs states of general quantum local Hamiltonians

(at least on regular lattices)

Approximate Quantum Hammersley-Clifford Theorem for 1D Systems



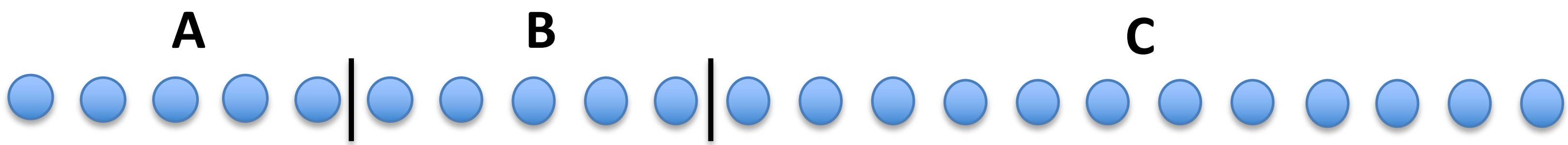
thm

1. Let H be a local Hamiltonian on n qubits. Then

$$I(A:C|B)_{\rho_T} \leq e^{-c'\sqrt{|B|}+e^{c/T}}$$

Gibbs state @ temperature T: $\rho_T := e^{-H/T}/Z$

Approximate Quantum Hammersley-Clifford Theorem for 1D Systems



thm

1. Let *H* be a local Hamiltonian on *n* qubits. Then

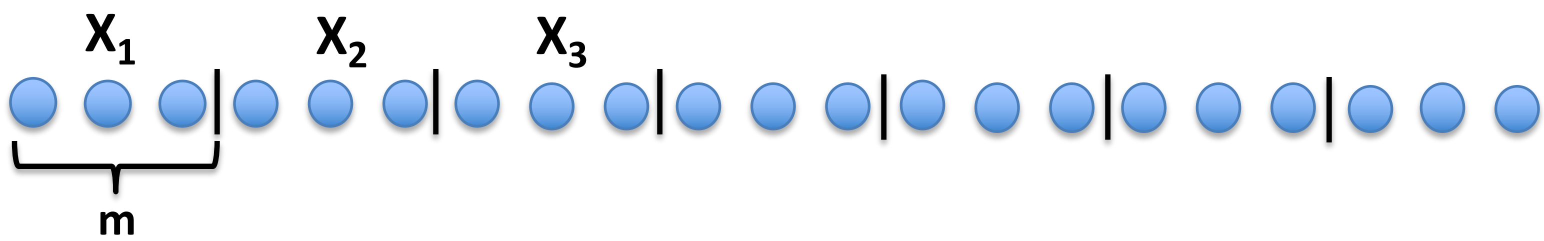
$$I(A:C|B)_{\rho_T} \leq e^{-c'\sqrt{|B|}+e^{c/T}}$$

2. Let $\rho_{1\ldots n}$ be a state on *n* qubits s.t. for every split ABC with |B| > m, $I(A:C|B) \leq \varepsilon$. Then

$$\min_{H \in \mathcal{H}_{2m}} S\left(\rho || e^{H}\right) \leq \varepsilon \frac{n}{m}$$

$$\mathcal{H}_{2m} := \{H : H = \sum_{k} H_{k,k+1},\ \forall k \ \mathrm{supp}(H_{k,k+1}) \leq 2m\}$$

Proof Part 2



Let $\sigma_{X_1 \ldots X_{\frac{n}{m}}}$ be the maximum entropy state s.t.

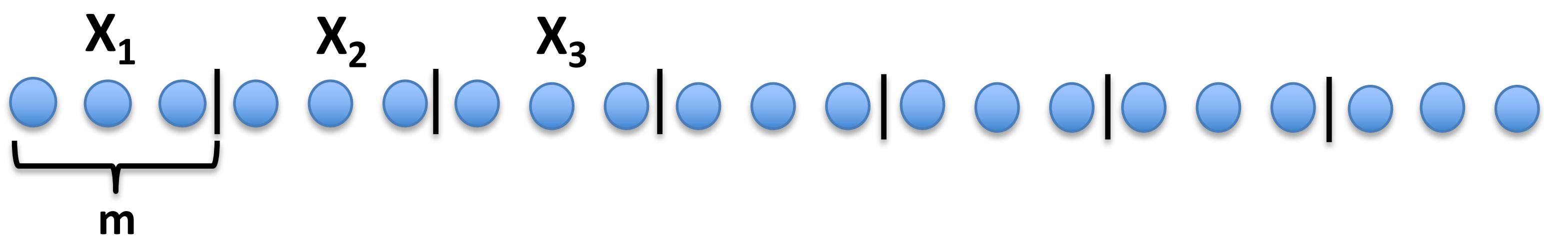
$$\sigma_{X_i, X_{i+1}} = \rho_{X_i, X_{i+1}} \quad \forall i \in [n/m]$$

Fact 1 (Jaynes' Principle): $\sigma = e^{\sum_k H_{X_k, X_{k+1}}}$

Fact 2
$$\begin{aligned} \min_{H \in \mathcal{H}_{2m}} S(\rho \| e^H / Z) &\leq -S(\rho) - \mathrm{tr}(\rho \log \sigma) \\ &= S(\sigma) - S(\rho) \end{aligned}$$

Let's show it's small

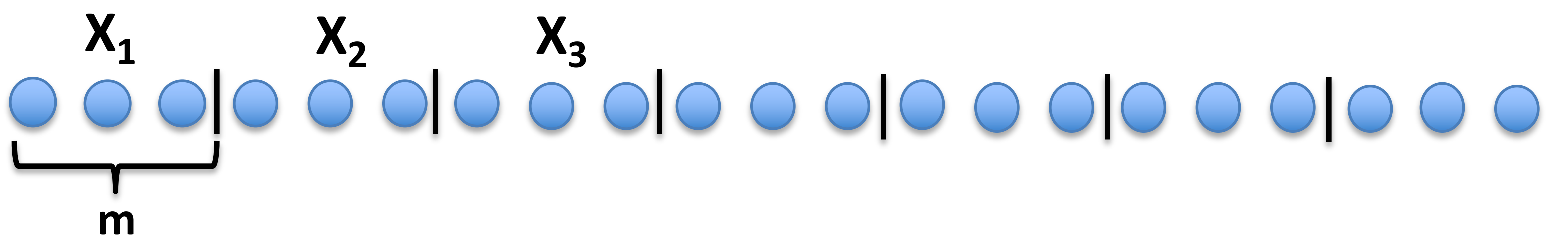
Proof Part 2



$$
\begin{aligned}
& S(X_1 \ldots X_{n/m})_\sigma \\
\leq \quad & S(X_1 X_2)_\sigma - S(X_2)_\sigma + S(X_2 \ldots X_{n/m})_\sigma
\end{aligned}
$$

SSA

Proof Part 2



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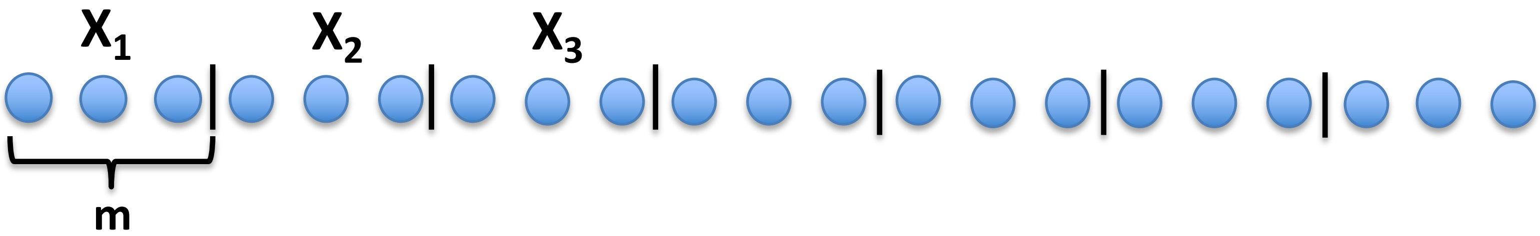
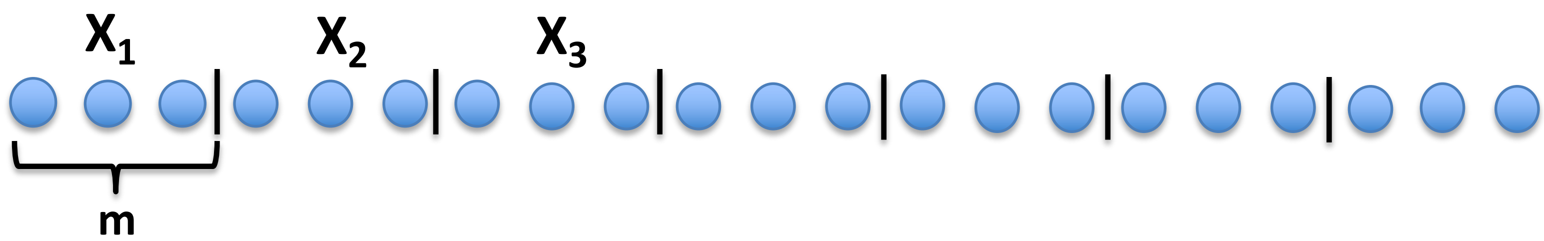
& S(X_1 \ldots X_{n/m})_\sigma \\
\leq \quad & S(X_1 X_2)_\sigma - S(X_2)_\sigma + S(X_2 \ldots X_{n/m})_\sigma \\
\leq \quad & S(X_1 X_2)_\sigma - S(X_2)_\sigma + S(X_2 X_3)_\sigma - S(X_3)_\sigma + S(X_3 \ldots X_{n/m})_\sigma
\end{aligned}
$$

Proof Part 2



$$
\begin{aligned}
& S(X_1 \ldots X_{n/m})_\sigma \\
\leq\ & S(X_1 X_2)_\sigma - S(X_2)_\sigma + S(X_2 \ldots X_{n/m})_\sigma \\
\leq\ & S(X_1 X_2)_\sigma - S(X_2)_\sigma + S(X_2 X_3)_\sigma - S(X_3)_\sigma + S(X_3 \ldots X_{n/m})_\sigma \\
\leq\ & \sum_i S(X_i X_{i+1})_\sigma - S(X_{i+1})_\sigma
\end{aligned}
$$

Proof Part 2



$$S(X_1 \ldots X_{n/m})_\sigma$$

$$\leq \quad S(X_1 X_2)_\sigma - S(X_2)_\sigma + S(X_2 \ldots X_{n/m})_\sigma$$

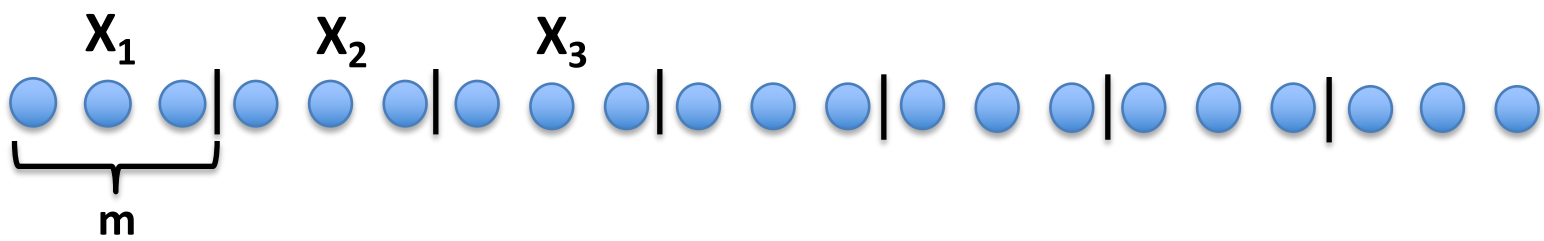
$$\leq \quad S(X_1 X_2)_\sigma - S(X_2)_\sigma + S(X_2 X_3)_\sigma - S(X_3)_\sigma + S(X_3 \ldots X_{n/m})_\sigma$$

$$\leq \quad \sum_i S(X_i X_{i+1})_\sigma - S(X_{i+1})_\sigma$$

$$= \quad \sum_i S(X_i X_{i+1})_\rho - S(X_{i+1})_\rho$$

Since $\sigma_{X_i, X_{i+1}} = \rho_{X_i, X_{i+1}} \quad \forall i \in [n/m]$

Proof Part 2



$$
\begin{aligned}
& S(X_1 \dots X_{n/m})_\sigma \\
\leq \quad & S(X_1 X_2)_\sigma - S(X_2)_\sigma + S(X_2 \dots X_{n/m})_\sigma \\
\leq \quad & S(X_1 X_2)_\sigma - S(X_2)_\sigma + S(X_2 X_3)_\sigma - S(X_3)_\sigma + S(X_3 \dots X_{n/m})_\sigma \\
\leq \quad & \sum_i S(X_i X_{i+1})_\sigma - S(X_{i+1})_\sigma \\
= \quad & \sum_i S(X_i X_{i+1})_\rho - S(X_{i+1})_\rho \\
\leq \quad & S(X_1 \dots X_{n/m})_\rho + \varepsilon \frac{n}{m}
\end{aligned}
$$

Since $I(X_i : X_{i+2} \dots X_{n/m} | X_{i+1}) \leq \varepsilon \ \forall i$

Proof Part 1

Recap: Let *H* be a local Hamiltonian on *n* qubits. Then

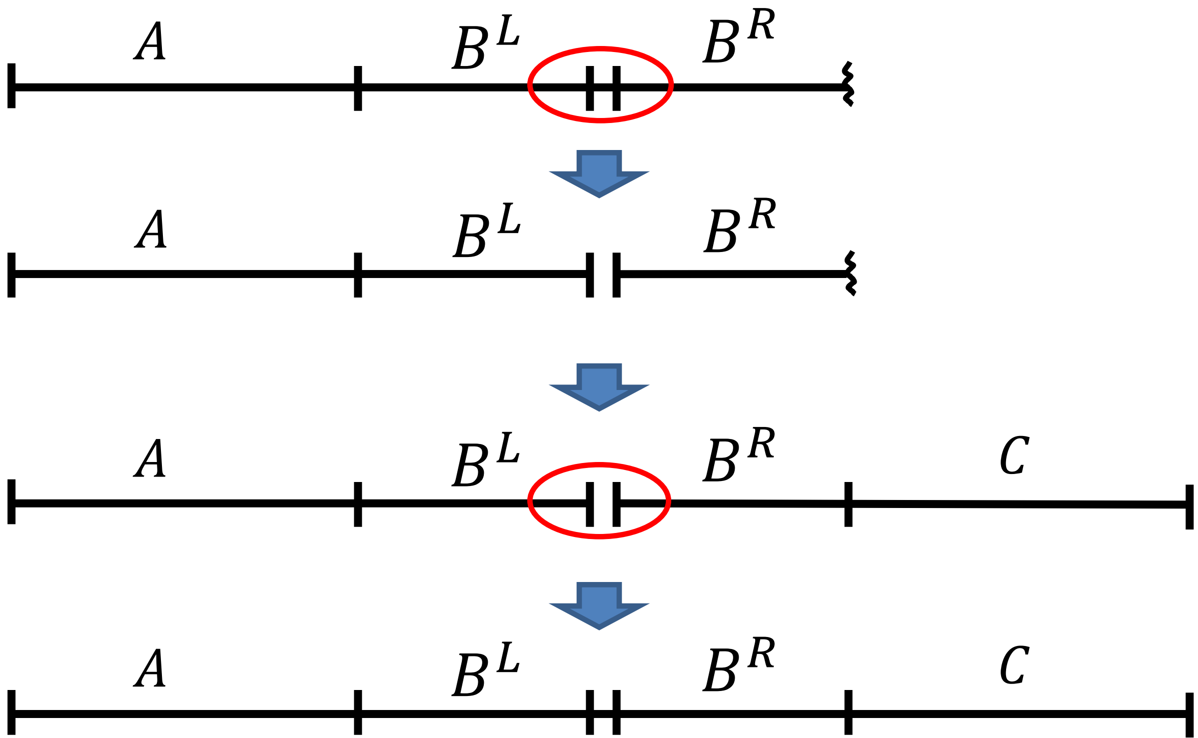
$$I(A:C|B)_{\rho_T} \leq e^{-c'\sqrt{|B|}+e^{c/T}}$$

We show there is a recovery channel from B to BC reconstructing the state on ABC from its reduction on AB.

Structure of Recovery Map

There exists an operator X_B such that

$$\rho^{H_{ABC}} \approx \mathrm{id}_A \otimes \kappa_{B\to \mathrm{BC}}\left(\rho_{AB}^{H_{ABC}}\right) = X_B\left(\mathrm{tr}_{B^R}\left[X_B^{-1}\rho_{AB}^{H_{ABC}}(X_B^{-1})^\dagger\right]\otimes \rho^{H_{B^R C}}\right)X_B^\dagger$$



Structure of Recovery Map

There exists an operator X_B such that

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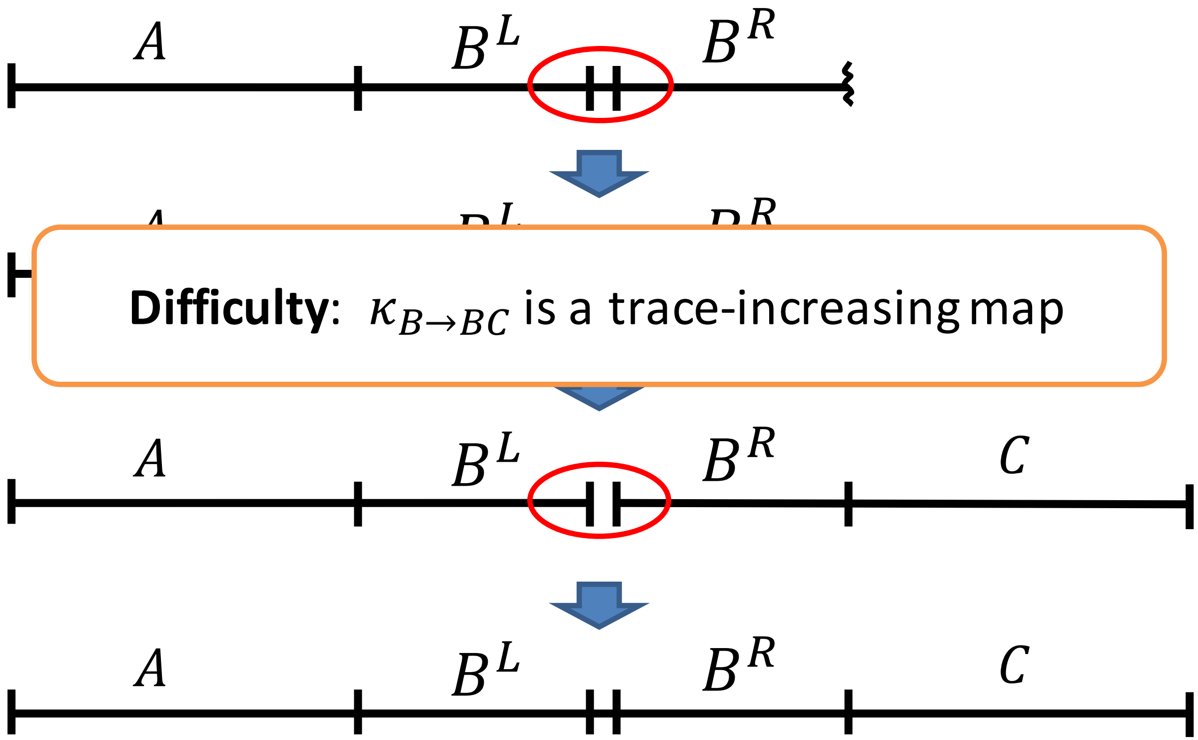
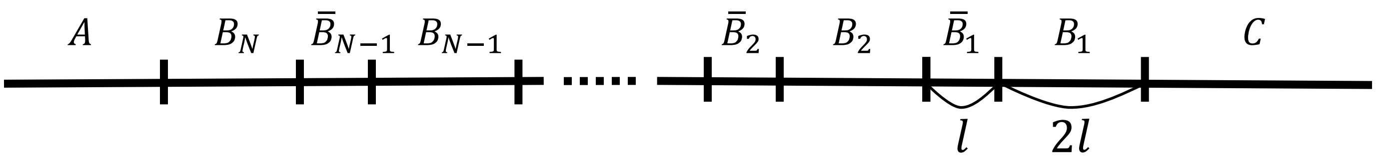
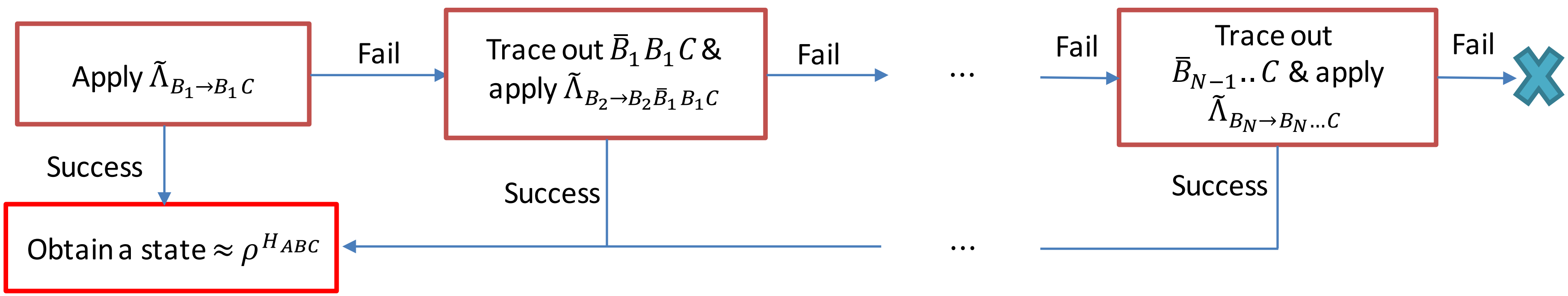


Repeat-until-success Method

We normalize $\kappa_{B\to BC}$ and define a CPTD-map $\tilde{\Lambda}_{B\to BC}$.
$\to$ Succeed to recover with a constant probability p.





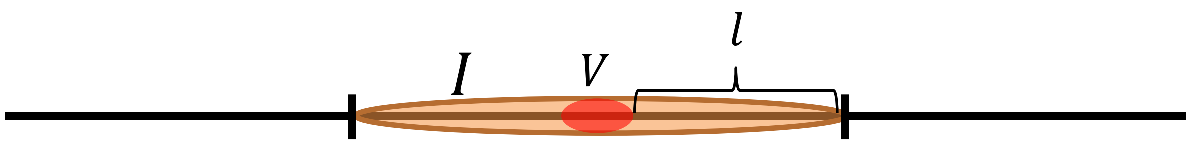
❑ Choose $N \sim l$ $\left(|B| = \mathcal{O}(l^2)\right)$.

$\to$Total error=Fail probability $(1-p)^l$ + approx. error $\mathcal{O}\left(e^{-\mathcal{O}(l)}\right) = \mathcal{O}\left(e^{-\mathcal{O}(l)}\right)$.

Locality of Perturbations

The key point in the proof:

For a short-ranged Hamiltonian H, the local perturbation to H only perturb the Gibbs state locally.



A useful lemma by Araki (Araki, '69)
For 1D Hamiltonian with short-range interaction H,

$$\|e^{H+V}e^{-H} - e^{H_I+V}e^{-H_I}\| \leq \mathcal{O}\left(e^{-\mathcal{O}(l)}\right)$$

$$e^{-\beta H} \to e^{-\beta(H+V)} \approx X_I e^{-\beta H} X_I^{\dagger}$$

$$X_I = e^{-\frac{\beta}{2}(H_I+V)} e^{\frac{\beta}{2}H_I} \leftarrow \text{Local}$$

Proof thm 1 part 2

We'll start with the second part. Recap:

Suppose $\gamma = 0$. Then there is a local $H = \sum_k H_{B_k, B_{k+1}}$ s.t.

$$\rho_{B_1 \ldots B_l} \approx \exp(H)/Z$$

Apply 1D approximate quantum Hammersley-Clifford thm to get

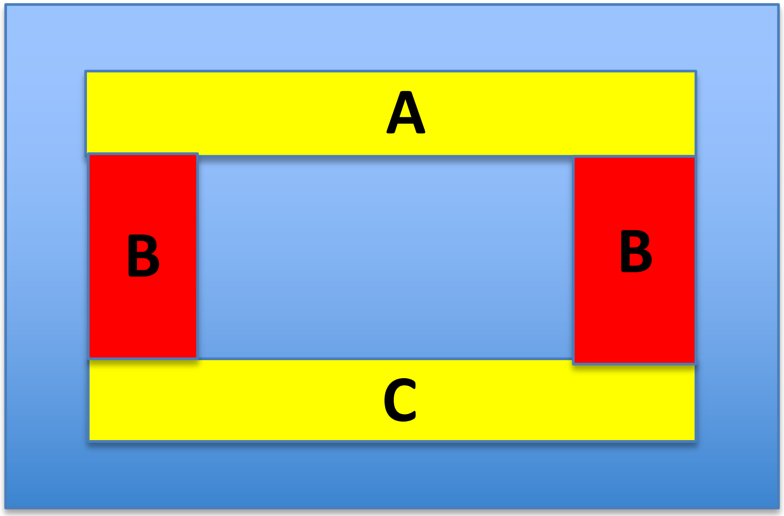
$$S(\rho_{B_1 \ldots B_l} \| \exp(H)/Z) \leq \frac{n}{m} e^{-cm}$$

With l = n/m. Choose m = O(log(n)) to make error small

Proof thm 1 part 1

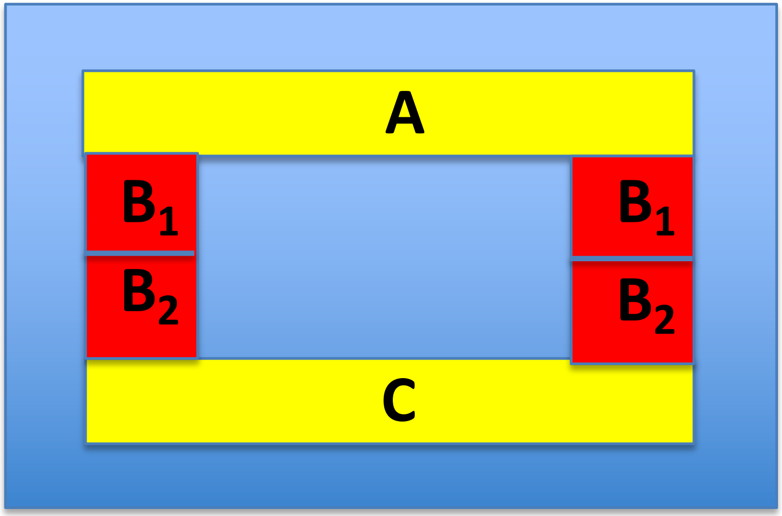
thm 1 Suppose $|\psi\rangle$ satisfies the area law assumption. Then

$$\begin{aligned} 2\gamma &\approx I(A:C|B) \\ &\approx \min_{H_{AB},H_{BC}} S(\rho_{ABC} \| \exp(H_{AB}+H_{BC})/Z) \end{aligned}$$



Proof thm 1 part 1

We follow the strategy of (Kato et al '15) for the zero-correlation length case



Area Law implies

$$I(A : B_2|B_1) \approx 0$$

$$I(C : B_1|B_2) \approx 0$$

By Fawzi-Renner Bound, there are channels $\Lambda : B_1 \to B_1 A$, $\Delta : B_2 \to B_2 C$ s.t.

$$\Lambda(\rho_{B_1 B_2}) \approx \rho_{AB_1B_2}, \quad \Delta(\rho_{B_1B_2}) \approx \rho_{B_1B_2C}$$

Proof thm 1 part 1

Define: $\sigma_{AB_1B_2C} := \Lambda^{B_1 \to B_1 A} \otimes \Delta^{B_2 \to B_2 C}(\rho_{B_1B_2})$

We have $\rho_{AB} \approx \sigma_{AB}, \ \rho_{BC} \approx \sigma_{BC}$

It follows that C can be reconstructed from B. Therefore

$$I(A:C|B)_\sigma \approx 0$$

Proof thm 1 part 1

Define: $\sigma_{AB_1B_2C} := \Lambda^{B_1 \to B_1 A} \otimes \Delta^{B_2 \to B_2 C}(\rho_{B_1 B_2})$

We have $\rho_{AB} \approx \sigma_{AB}, \; \rho_{BC} \approx \sigma_{BC}$

It follows that C can be reconstructed from B. Therefore

$$I(A:C|B)_\sigma \approx 0$$

Since

$$I(A:C|B)_\sigma = S(\sigma_{ABC} \| \exp(\log(\sigma_{AB})) + \log(\sigma_{BC})) - \log(\sigma_B)))$$

$\pi \approx \sigma$ with

$$\pi := \exp(\log(\sigma_{AB}) + \log(\sigma_{BC}) - \log(\sigma_B))/\mathrm{tr}(\ldots)$$

So $I(A:C|B)_\pi \approx 0$

Proof thm 1 part 1

Since $I(A:C|B)_\pi \approx 0$

$$
\begin{aligned}
S(ABC)_\pi &\approx S(AB)_\pi + S(BC)_\pi - S(B)_\pi \\
&\approx S(AB)_\rho + S(BC)_\rho - S(B)_\rho \\
&= S(ABC)_\rho + I(A:C|B)_\rho
\end{aligned}
$$

Let R_2 be the set of Gibbs states of Hamiltonians $H = H_{AB} + H_{BC}$. Then

$$
\begin{aligned}
\min_{\nu \in R_2} S(\rho \| \nu) &= \min_{\nu \in R_2} -S(\rho) - \mathrm{tr}(\rho \log \nu) \\
&\approx I(A:C|B)_\rho + \min_{\nu \in R_2} -S(\pi) - \mathrm{tr}(\rho \log \nu) \\
&\approx I(A:C|B)_\rho + \min_{\nu \in R_2} -S(\pi) - \mathrm{tr}(\pi \log \nu) \\
&= I(A:C|B)_\rho
\end{aligned}
$$

Open Problems

- What happens in dim bigger than 2?
- Can we prove the approximate Markov property for general quantum states?
- Can we prove the converse, i.e. that approximate quantum Markov Networks are approximately thermal?
- Are two copies of the entanglement spectrum necessary to get a local boundary model?

Thanks!