Equivalence of Ensembles from Finite Correlation Length

Fernando G.S.L. Brandão
Microsoft Research
based on joint work with
Marcus Cramer
University of Ulm

Quantum Spin Systems, Recent Advances, Cergy-Pontoise, 2015
Quantum Information vs Quantum Statistical Mechanics

- Microcanonical typicality vs measure concentration
- Thermodynamics as a resource theory
- Area laws and tensor networks for thermal states
- Generic thermalization vs quantum pseudo-randomness
- ....
Quantum Information vs Quantum Statistical Mechanics

Ex.

- Microcanonical typicality vs measure concentration
- Thermodynamics as a resource theory
- Area laws and tensor networks for thermal states
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This talk: Equivalence of microcanonical and canonical ensembles from finite correlation length (using ideas from quantum information theory)
Microcanonical vs Canonical Ensembles

Given a Hamiltonian of $n$ particles:

$$H = \sum_j H_j = \sum_k E_k |E_k\rangle \langle E_k|$$

$C^2$
Microcanonical vs Canonical Ensembles

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Microcanonical:

$$\langle X \rangle_{mc,e} := \text{tr}(\tau_e X)$$

$$\tau_{e,\delta} := \frac{1}{|M_{e,\delta}|} \sum_{k \in M_{e,\delta}} |E_k\rangle \langle E_k|$$

$$M_{e,\delta} = \{ k : |E_k - en| \leq \delta \sqrt{n} \}$$
Microcanonical vs Canonical Ensembles

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**Microcanonical:**

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\[
M_{e,\delta} = \{k : |E_k - en| \leq \delta \sqrt{n}\}
\]

**Canonical:**

\[
\langle X \rangle_{c,\beta} := \text{tr}(\rho_\beta X)
\]

\[
\rho_\beta := e^{-\beta H} / Z
\]
Microcanonical vs Canonical Ensembles

When should we use each?

**Micro:** System in isolation  
**Macro:** System in equilibrium with a heat bath at temperature 1/β

Can derive macro in S from micro in S+E. Justified whenever the interaction of system and environment is very weak.
Microcanonical vs Canonical Ensembles

When should we use each?

**Micro:** System in isolation
**Macro:** System in equilibrium with a heat bath at temperature $1/\beta$

What if we are only interested in expectation values of local observables?

Is the system an environment for itself?

$$\langle X \rangle_{mc,e(\beta)} \approx \langle X \rangle_{c,\beta} \text{ for } e(\beta) = tr(H\rho_\beta)/n \text{ and local } X?$$
Equivalence of Ensembles for non-critical systems

**Gibbs 1902**: For the average square of the anomalies of the energy, we find an expression which vanishes in comparison to the square of the average energy, when the number of degrees of freedom is indefinitely increased. An ensemble of systems in which the number of degrees of freedom is of the same order of magnitude as the number of molecules in the bodies with which we experiment, if distributed canonically, would therefore appear to human observation as an ensemble of systems in which all have the same energy.

\[ C(T) = \text{var}(E) = O(n) \]

Fluctuations of energy of order \( O(n^{1/2}) \) only

To simplistic: microcanonical and canonical states are almost orthogonal for large systems.
Previous Results

- Equivalence for Classical System
  (Ruelle ‘69, Aizenman, Goldstein, Lebowitz ’78, ...)

- Dependence on Correlation Length
  (e.g. 2D Ising model)  (Deseremo ’04)

- Equivalence for local observables in infinite lattices in
  the “unique phase region” (i.e. only one KMS state).

\[
\lim_{n \to \infty} \left\| \text{tr}_{\Lambda_n} \Lambda \left( \tau_{u(T), o(\sqrt{N})}^{\Lambda_n} \right) - \text{tr}_{\Lambda_n} \Lambda \left( \rho_T^{\Lambda_n} \right) \right\|_1 = 0
\]

**thm** (Lima ‘72; Muller, Adlam, Masanes, Wiebe ‘13)
Sequence of translation-invariant Hamiltonians \( H_{\Lambda_n} \) on finite volume \( \Lambda_n \) with unique KMS state:
Main Result

**thm** Let $H$ be a Hamiltonian of $n$ particles on a $d$-dimensional lattice. Let $\theta$ be such that $\rho_{\theta}$ has a correlation length $\xi$.

Fix $\delta$ s.t. $c_1 \log^{2d}(n)n^{-1/2} \leq \delta \leq c_2$.

Then for most regions $A$ of size at most $c_3(\varepsilon n \xi^{-d})\frac{1}{d+1}$

$$\|\text{tr} A^c(\tau_{\varepsilon(\beta)}) - \text{tr} A^c(\rho_{\beta})\|_1 \leq \varepsilon$$

**Correlation length $\xi$:** For all $X, Z$

$$\text{cov}(X, Z)_{\rho_{\beta}} \leq 2^{-\text{dist}(X,Z)/\xi}$$

$$\text{cov}(X, Z)_{\rho_{\beta}} := \langle XZ \rangle_{\rho_{\beta}} - \langle X \rangle_{\rho_{\beta}} \langle Z \rangle_{\rho_{\beta}}$$
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$$\| \text{tr}_{A^c} \left( \tau_{\epsilon \beta} \right) - \text{tr}_{A^c} (\rho_\beta) \|_1 \leq \epsilon$$

- It gives explicit finite size bounds
- Works for non-translational invariant systems
- It’s based on finite correlation length (instead unique KMS state)
- Shows equivalence works even under rather small energy spread – of order $O(\log^{2d}(n))$
Main Result

**thm** Let $H$ be a Hamiltonian of $n$ particles on a $d$-dimensional lattice. Let $\beta$ be such that $\rho_\beta$ has a correlation length $\xi$.

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Then for most regions $A$ of size at most $c_3(\varepsilon n \xi^{-d})^{\frac{1}{d+1}}$

$$\left\| \text{tr}_{A^c} \left( \tau_{e(\beta)} \right) - \text{tr}_{A^c} \left( \rho_\beta \right) \right\|_1 \leq \varepsilon$$

- Extension of result covers *beyond microcanonical state*:

**Ex 1.** Any state $\tau$ in $\text{span}\{ |E_k\rangle \}_{k \in M_e, \delta}$ with entropy

$$S(\tau) \geq \log(|M_{e, \delta}|) - c(\varepsilon N \xi^{-d})^{\frac{1}{d+1}}$$

**Ex 2.** A generic state in $\text{span}\{ |E_k\rangle \}_{k \in M_e, \delta}$ following (Popescu, Short, Winter ‘06)
Let sequence $H_{\Lambda_n}$ on finite volumes $\Lambda_n$.

Free energy density:

$$f(T) := \inf \{ f_T(\omega) : \omega \text{ translation-invariant} \}$$

with $f_T(\omega) = u(\omega) - T s(\omega)$.

Can show (Simons '93)

$$\lim_{n \to \infty} \frac{1}{n} F_T(\tau_{u(T), o(n)}^{\Lambda_n}) = \lim_{n \to \infty} \frac{1}{n} F_T(\rho_T^{\Lambda_n})$$

One phase region: unique minimizer $\omega$ for $f(T)$. Therefore:

$$\lim_{n \to \infty} \left\| \text{tr}_{\Lambda_n \backslash \Lambda}(\tau_{u(T), o(\sqrt{N})}^{\Lambda_n}) - \text{tr}_{\Lambda_n \backslash \Lambda}(\rho_T^{\Lambda_n}) \right\|_1 = 0$$
The Muller *et al* Argument vs Ours

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\]

Lemma 1: Let $\tau$, $\rho$ be states of $N$ qubits on $d$ dims. Let $\rho$ have correlation length $\xi$. If

\[
S(\tau \| \rho) \leq c \log^{2d}(n)
\]

for most regions of size $c'(\varepsilon n \xi^{-d})^{\frac{1}{d+1}}$

\[
\|\tau_C - \rho_C\|_1 \leq \sqrt{\varepsilon}
\]

As

\[
T S(\tau \| \rho_T) = F_T(\tau) - F_T(\rho_T)
\]

applying the Lemma to $\rho = \rho_T$, gives a finite size analogue of unique minimizer for $f(T)$
The Muller et al Argument vs Ours

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S(\tau \| \rho) \leq c \log^{2d}(n)
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for most regions of size $\varepsilon n \xi^{-d} \frac{1}{d+1}$

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\| \tau_C - \rho_C \|_1 \leq \sqrt{\varepsilon}
\]

**Lemma 2:** If $\rho_T$ has finite corr. length

\[
S(\tau_{u(T)}, \delta \| \rho_T) \leq c \log^{2d}(n)
\]

Finite size analogue of convergence of free energy densities.
Lemma 2: If $\rho_T$ has finite corr. length:  

$$S(\tau_u(T),\delta\|\rho_T) \leq c \log^{2d}(n)$$

Follow from the following new Berry-Esseen like thm for quantum lattice systems:

What’s the distribution of energy (given by Ham $H$) in a quantum state $\rho$?

If $\rho$ is product and $H$ is non-interacting, it’s given by a sum of i.i.d. variables.

Central limit: Goes to a normal distribution

Berry-Esseen: Error decays as $O(n^{-1/2})$

What if $H$ is interacting, but local?

(Goderis, Vetz ’89, Hartmann, Mahler, Hess ’04)

Quantum central limit: Goes to a normal distribution

Quantum Berry-Esseen?
Quantum Berry-Esseen

**thm** Let $H$ be a local Hamiltonian in $d$ dims and $\rho$ a state with correlation length $\xi$. Let

$$F(x) = \sum_{k:E_k \leq x} \langle k | \rho | k \rangle, \quad \mu = \text{tr}(H \rho), \quad \sigma^2 = \text{tr}(\rho(H - \mu)^2)$$

and

$$G(x) = \frac{1}{\sqrt{2\pi \sigma^2}} \int_{-\infty}^{x} dy e^{-\frac{(y-\mu)^2}{2\sigma^2}}.$$

Then

$$\sup_x |F(x) - G(x)| \leq \Delta(\xi) \frac{\log^{2d}(n)}{\sqrt{n}}$$

Follows proofs of classical Berry-Esseen thm for weakly correlated variables by (Sunklodas ‘84) and (Tikhomirov ‘80).
Proof of Lemma 2

**Lemma 2:** If $\rho_T$ has finite corr. length: \[ S(\tau_u(T), \delta \| \rho_T) \leq c \log^{2d}(n) \]

Set $N := n$.

We prove it for $\delta = c \log^{2d}(N)N^{-1/2}$. Let $Z(T, e, \delta) := \sum_{k \in M_e, \delta} e^{-E_k/T}$

\[
\frac{Z(T, e, \delta)}{Z(T)} = \text{tr} \left( \rho_T \sum_{k \in M_e, \delta} |E_k\rangle\langle E_k| \right)
\]

\[
= F(eN + \delta \sqrt{N}) - F(eN - \delta \sqrt{N})
\]

\[
\geq G(eN + \delta \sqrt{N}) - G(eN - \delta \sqrt{N}) - 2\Delta \frac{\log^{2d}(N)}{\sqrt{N}} \geq C \frac{\log^{2d}(N)}{\sqrt{N}}
\]

Using: \[ |M_{e, \delta}|^{-1} \leq e^{2c \log^{2d}(N)/T} e^{-E_k/T} Z(T, e, \delta)^{-1} \]

\[
\tau_{e, \delta} \leq \sqrt{Ne^{2c \log^{2d}(N)/T}} \frac{1}{Z(T)} \sum_{k \in M_{e, \delta}} e^{-E_k/T} |E_k\rangle\langle E_k| \leq \sqrt{Ne^{2c \log^{2d}(N)/T}} \rho_T
\]
Lemma 1: Let $\tau, \rho$ be states of $N$ qubits in $d$ dims. Let $\rho$ have correlation length $\xi$. If

$$S(\tau||\rho) \leq c \log^{2d}(N)$$

for most regions of size $c'(\varepsilon N \xi^{-d})^{\frac{1}{d+1}}$, $\|\tau_C - \rho_C\|_1 \leq \sqrt{\varepsilon}$
Relative Entropies

Relative Entropy: \( S(\tau\|\rho) = \text{tr}(\tau(\log \tau - \log \rho)) \)

Smooth Rel. Ent.: \( S^\varepsilon(\tau\|\rho) = \min_{\tilde{\tau} \in B_\varepsilon(\tau)} S(\tilde{\tau}\|\rho), \ B_\varepsilon(\tau) := \{\tilde{\tau} : \|\tau - \tilde{\tau}\|_1 \leq \varepsilon\} \)

Max Rel. Ent.: \( S_{\text{max}}(\tau\|\rho) := \{\min \lambda : \tau \leq 2^\lambda \rho\} \)

Smooth Max Rel. Ent.: \( S^\varepsilon_{\text{max}}(\tau\|\rho) = \min_{\tilde{\tau} \in B_\varepsilon(\tau)} S_{\text{max}}(\tilde{\tau}\|\rho) \)

Quantum Substate:

\[
S^2\sqrt{\varepsilon}(\tau\|\rho) \leq S^\varepsilon_{\text{max}}(\tau\|\rho) \leq \frac{S(\tau\|\rho) + 1}{\varepsilon} + \log \left(\frac{1}{1 - \varepsilon}\right)
\]

Data Processing: \( S(\rho_{AB}\|\sigma_{AB}) \geq S(\rho_A\|\sigma_A) \)

Pinsker’s inequality: \( S(\rho\|\sigma) \geq \frac{1}{2 \ln(2)} \|\rho - \sigma\|_1^2 \)
Proof of Lemma 1 for 1 dim

Data Processing + substate thm:
\[ S_{\text{max}}^{2\varepsilon}(\tau_{A_1} \ldots \tau_{A_m} \parallel \rho_{A_1} \ldots A_m) \leq \left( S(\tau \parallel \rho) + 1 \right) / \varepsilon \]

Corr. length \( \xi \):
\[ \| \rho_{A_1} \ldots A_m - \rho_{A_1} \otimes \ldots \otimes \rho_{A_m} \|_1 \leq m 2^{2l} 2^{-r/\xi} \]

(Datta, Renner ‘08)
\[ S_{\text{max}}^{\sqrt{8\kappa}}(\tau \parallel \tilde{\rho}) \leq S_{\text{max}}(\tau \parallel \rho) - \log(1 - \kappa) \]

where \( \kappa := 2^{S_{\text{max}}(\tau \parallel \rho)} \| \tilde{\rho} - \rho \|_1 \)

Then
\[ S_{\text{max}}^{\varepsilon'}(\tau_{A_1} \ldots A_m \parallel \rho_{A_1} \otimes \ldots \otimes \rho_{A_m}) \leq c_1 \log^{2d}(N), \]

with \( \varepsilon' = 2\sqrt{\varepsilon} + 2^{c_3 \log^{2d}(N)} 2^{-c_2 l} \).
Proof of Lemma 1 for 1 dim

\begin{align*}
S_{\text{max}}^{\varepsilon'}(\tau_{A_1} \cdots A_m || \rho_{A_1} \otimes \cdots \otimes \rho_{A_m}) & \leq c_1 \log^{2d}(N), \\
\varepsilon' &= 2\sqrt{\varepsilon} + 2^{c_3 \log^{2d}(N)} 2^{-c_2 l} \\
\text{By subadditivity: } \sum_{i=1}^{m} S^{\varepsilon'}(\tau_{A_i} || \rho_{A_i}) & \leq c_1 \log^{2d}(N) \\
\text{By Pinsker’s inequality: } \sqrt{S^{\varepsilon'}(\tau_{A_i} || \rho_{A_i})} & \geq \| \tau_{A_i} - \rho_{A_i} \|_1 / \ln(4) - \varepsilon'
\end{align*}

Since \( m = n / ((4\xi + 1)l) \), \( \mathbb{E}_i \| \tau_{A_i} - \rho_{A_i} \|_1 \leq c' \left( l \frac{\log^{2d}(N)}{N} \right)^{1/2} \)
Open Questions

• Are the ensembles equivalent up to sizes $\Omega(n)$?

• What happens in a symmetry-broken phase or at criticality?

• Can we prove a quantum Berry-Esseen thm without polylog$(N)$ terms?

• Are there more applications of quantum Berry-Esseen thm?

Thanks!