Quantum Information, Entanglement, and Many-Body Physics

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Quantum Information Theory

**Goal:** Lay down the theory for future quantum-based technology (quantum computers, quantum cryptography, ...)

- Quantum Comm.
- Entanglement theory
- Q. error correc. + FT
- Quantum comp.
- Quantum complex. theo.
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Ultimate limits to information transmission

Entanglement as a resource
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Entanglement as a resource
QIT Connections

QIT

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QIT Connections

Condensed Matter
- Strongly corr. systems
- Topological order
- Spin glasses

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HEP/GR
Topolog. q. field theo.
Black hole physics
Holography

StatMech
Thermalization
Thermo@nano scale
Quantum-to-Classical Transition

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Exper. Phys.
- Ion traps, linear optics,
  optical lattices, cQED,
  superconduc. devices,
  many more
This Talk

**Goal:** give examples of these connections:

1. **Entanglement in many-body systems**
   - area law in 1D from finite correlation length
   - product-state approximation to low-energy states in high dimensions

2. **(time permitting) Quantum-to-Classical Transition**
   - show that distributed quantum information becomes classical (quantum Darwinism)
Entanglement in quantum information science is a resource (teleportation, quantum key distribution, metrology, ...)

Ex. EPR pair $|\phi^+\rangle = (|0, 0\rangle + |1, 1\rangle)/\sqrt{2}$

How to quantify it?

Bipartite Pure State Entanglement

Given $|\psi\rangle_{AB}$, its entropy of entanglement is

$$E(|\psi\rangle_{AB}) := S(\rho_A) = S(\rho_B)$$

Reduced State: $\rho_A := \text{tr}_B(|\psi\rangle\langle\psi|_{AB})$

Entropy: $S(\rho) = -\text{tr}(\rho \log \rho)$ (Rényi Entropies: $S_\alpha(\rho) := \frac{1}{1-\alpha} \log \text{tr}(\rho^\alpha)$)
Entanglement in Many-Body Systems

A quantum state $\psi$ of $n$ qubits is a vector in $\mathbb{C}^2 \otimes n \cong \mathbb{C}^{2^n}$

$$|\psi\rangle = \sum_{i_1, \ldots, i_n} c_{i_1, \ldots, i_n} |i_1, \ldots, i_n\rangle$$

For almost every state $\psi$, $S(X)_{\psi} \approx |X|$ (for any $X$ with $|X| < n/2$)

$|X| := \# \text{qubits in } X$
Def: $\psi$ satisfies an area law if there is $c > 0$ s.t. for every region $X$,

$$S(X) \leq c \text{Area}(X)$$

Entanglement is Holographic
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$$S(X) \leq c \text{Area}(X)$$

Entanglement is Holographic

When do we expect an area law?

Low-energy states of many-body local models
Def: $\psi$ satisfies an area law if there is $c > 0$ s.t. for every region $X$,
\[ S(X) \leq c \text{Area}(X) \]

Entanglement is Holographic

When do we expect an area law?

Low-energy states of many-body local models

(Bombeli et al '86) massless free scalar field (connection to Bekenstein-Hawking entropy)
(Vidal et al '03; Plenio et al '05, ...) XY model, quasi-free bosonic and fermionic models, ...
(Holzhey et al '94; Calabrese, Cardy '04) critical systems described by CFT (log correction)

(Aharonov et al '09; Irani '10) 1D model with volume scaling of entanglement entropy!
Why Area Law is Interesting?

- Connection to **Holography**.

- Interesting to study entanglement in physical states with an eye on **quantum information processing**.

- Area law appears to be connected to our ability to write-down **simple Ansatzes** for the quantum state. (e.g. tensor-network states)

This is known rigorously in 1D:
Matrix Product States

(Fannes, Nachtergaele, Werner ’92; Affleck, Kennedy, Lieb, Tasaki ‘87)

\[
|\psi\rangle_{1,...,n} = \sum_{i_1=1}^{2} \cdots \sum_{i_n=1}^{2} \text{tr} \left( A_{i_1}^{[1]} \cdots A_{i_n}^{[n]} \right) |i_1,...,i_n\rangle, \ A_j^{[l]} \in \text{Mat}(D,D)
\]

D : bond dimension

\[
D^{-1/2} \sum_{i=1}^{D} |i,i\rangle
\]

\[
A^i := \sum_{i,\alpha,\beta} A_{\alpha,\beta}^{[i]} |i\rangle \langle \alpha, \beta |
\]

- Only \(nD^2\) parameters.
- Local expectation values computed in \(nD^3\) time
- Variational class of states for powerful DMRG
- Generalization of product states (MPS with \(D=1\))
MPS ↔ Area Law

- For MPS, $S(\rho_X) \leq \log(D)$

- (Vidal ’03; Verstraete, Cirac ‘05)
  
  If $\psi$ satisfies $S(\rho_X) \leq \log(D)$ for all $X$, then it has a MPS description of bond dim. $D$

(obs: must use Renyi entropies)
Correlation Length

Correlation Function:

\[
\text{cor}(X : Z)_\rho := \max_{M \in X, N \in Y} \text{tr}((M \otimes N)(\rho_{XZ} - \rho_X \otimes \rho_Z))
\] 

\[
\|M\|, \|N\| \leq 1
\]

For pure state \(\Psi\)

\[
\text{cor}(X : Z)_\psi := \max_{M \in X, N \in Y} \langle \psi | M \otimes N | \psi \rangle - \langle \psi | M | \psi \rangle \langle \psi | N | \psi \rangle
\]

Correlation Length:

\(\psi\) has correlation length \(\xi\) if for every regions \(X, Z\):

\[
\text{cor}(X : Z)_\psi \leq 2^{-\text{dist}(X, Z) / \xi}
\]

\[
\text{dist}(X, Z) := \min_{x \in X, z \in Z} \text{dist}(x, z)
\]
When there is a finite correlation length?

(Araki ‘69) In 1D at any finite temperature $T$
(for $\rho = e^{-H/T}/Z$; $\xi = O(1/T)$)

(Hastings ‘04) In any dim at zero temperature for gapped models
(for groundstates; $\xi = O(1/gap)$)

(Hastings ‘11; Hamza et al ‘12; ...) In any dim for models with mobility gap
(many-body localization)

(Kliesch et al ‘13) In any dim at large enough $T$

(Kastoryano et al ‘12) Steady-state of rapidly-mixing dissipative processes (e.g. gapped Liovillians)
Area Law from Correlation Length?

\[ \text{Ent}(X : X^c) \approx \sum_{i \in X^c} \text{Ent}(X : X_i) \]
\[ \approx \sum_{i \in X^c} 2^{-\text{dist}(X,i) / \xi} \]
\[ = O(\text{Area}(X) \xi) \]
Area Law from Correlation Length?

\[ \psi \]

\[ \text{Ent}(X : X^c) \approx \sum_{i \in X^c} \text{Ent}(X : X_i) \]
\[ \approx \sum_{i \in X^c} 2^{-\text{dist}(X,i)/\xi} \]
\[ = O(\text{Area}(X)\xi) \]

That’s incorrect!

\textbf{Ex.} For almost every } n \text{ qubit state, } S(X) \approx \text{vol}(X) \text{ but for all } i \text{ in } X^c, \quad \|\rho_{XX_i} - \rho_X \otimes \rho_{X_i}\|_1 \leq 2^{-cn} \text{ Entanglement can be non-locally encoded (e.g. QECC, Topological Order)}
Area Law from Correlation Length?

Suppose $\rho_{XZ} = \rho_X \otimes \rho_Z$. 

$|\psi\rangle_{XYZ}$
Area Law from Correlation Length?

Suppose $\rho_{XZ} = \rho_X \otimes \rho_Z$.

Then $|\psi\rangle_{XYZ} = (U_{XZ\rightarrow Y} \otimes I_{XZ}) |\eta\rangle_{XX} \otimes |\nu\rangle_{ZZ}$

$X$ is only entangled with $Y$
Area Law from Correlation Length?

Suppose $\rho_{XZ} = \rho_X \otimes \rho_Z$.

Then $|\psi\rangle_{XYZ} = (U_{XZ \rightarrow Y} \otimes I_{XZ}) |\eta\rangle_{XX} \otimes |\nu\rangle_{ZZ}$

$X$ is only entangled with $Y$

But there are states (data hiding, quantum expanders) for which

$\text{Cor}(X:Y) \leq 2^{-l}$ and $\|\rho_{XZ} - \rho_X \otimes \rho_Z\|_1 \approx 2$

Small correlations in a fixed partition doesn’t mean anything
Area Law in 1D?

Gapped Ham \(\rightarrow\) Finite Correlation Length \(\rightarrow\) Area Law

(Hastings ’04)  ???  Vidal ’03

MPS Representation
**Area Law in 1D**

\[ (\text{Hastings '07}) \]

- **Gapped Ham** (Hastings '04)
- **Finite Correlation Length**
- **Area Law**
- **MPS Representation** (Vidal '03)

**thm (Hastings '07)** For \( H \) with spectral gap \( \Delta \) and unique groundstate \( \Psi_0 \), for every region \( X \),

\[ S(X)_{\Psi} \leq \exp(c / \Delta) \]

(Arad, Kitaev, Landau, Vazirani '12) \( S(X)_{\Psi} \leq c / \Delta \)
“Interestingly, states that are defined by quantum expanders can have exponentially decaying correlations and still have large entanglement, as has been proven in (...)”
Correlation Length vs Entanglement I

**thm 1** (B., Horodecki ‘12) Let $|\psi\rangle_1,...,n$ be a quantum state in 1D with correlation length $\xi$. Then for every X,

$$S(X)_\psi \leq 2^{c\xi}$$

- The statement is only about quantum states, no Hamiltonian involved.
- Applies to gapless models with finite correlation length e.g. systems with mobility gap (many-body localization)
**Correlation Length vs Entanglement II**

**thm 2** Let \( \{ |\psi_k\rangle \}_{k=1}^{N} \) be quantum states in 1D with correlation length \( \xi \). Then for every \( k \) and \( X \),

\[
S(X)\psi_k \leq 2^{c_1\xi} + c_2 \log(N)
\]

Applies to 1D gapped Hamiltonians with degenerate groundstates.
**thm 2** Let \( \{ |\psi_k\rangle\}_{k=1}^N \) be quantum states in 1D with correlation length \( \xi \). Then for every \( k \) and \( X \),

\[
S(X)_{\psi_k} \leq 2^{c_1 \xi} + c_2 \log(N)
\]

**Def:** \( \{ |\psi_k\rangle\}_{k=1}^N \) have correlation length \( \xi \) if for every \( i \) and regions \( X, Z \):

\[
\text{cor}_P(X : Z)_{\psi_i} \leq 2^{-\text{dist}(X, Z)/\xi}
\]

with \( P = \sum_i |\psi_i\rangle\langle\psi_i| \)

and \( \text{cor}_P(X : Z)_{\psi_i} := \max_{M \in X, N \in Z} \langle \psi_i | M \otimes N | \psi_i \rangle - \langle \psi_i | MPN | \psi_i \rangle \)

**Applies to** 1D gapped Hamiltonians with degenerate groundstates.
**Application: Adiabatic Quantum Computing in 1D**

Quantum computing by dragging: Prepare $\psi(0)$ and adiabatically change $H(s)$ to obtain $\psi(s_f)$

$$H(0) \quad \psi_0 \quad H(s) \quad \psi_s \quad H(s_f)$$

$$H(s)\psi_s = E_{0,s}\psi_s$$

$$\Delta := \min \Delta(s)$$

(Aharonov et al ‘08)

Universal in 1D with unique groundstate and $\Delta > 1/poly(n)$

(Hastings ‘09)

Non-universal in 1D with *unique* groundstate and constant $\Delta$

(Bacon, Flammia ‘10)

Universal in 1D with *exponentially* many groundstates and constant $\Delta$

**cor:** Adiabatic computation using 1D gapped $H(s)$ with $N$ groundstates can be simulated classically in time $\exp(N)\text{poly}(n)$
**thm 3** Let $\rho_1, \ldots, \rho_n$ be a mixed quantum state in 1D with correlation length $\xi$. Let $|\psi\rangle := \rho^{1/2} \otimes I |\phi^+\rangle_{S\bar{S}}$. Then

$$S(X\bar{X})_\psi \leq 2^{c\xi}$$

- Implies area law for thermal states at any non-zero temperature
Summing Up

Area law always holds in 1D whenever there is a finite correlation length:

- Groundstates (unique or degenerate) of gapped models
- Groundstates of models with mobility gap (many-body localization)
- Thermal states at any non-zero temperature
- Steady-state of gapped dissipative dynamics

Implies that in all such cases the state has an efficient classical parametrization as a MPS

(Useful for numerics – e.g. DMRG. Limitations for quantum information processing)
We want to bound the entropy of $X$ using the fact the correlation length of the state is finite.

Need to relate entropy to correlations.
Entanglement Distillation

Consists of extracting EPR pairs from bipartite entangled states by Local Operations and Classical Communication (LOCC)

Central task in quantum information processing for distributing entanglement over large distances (e.g. entanglement repeater)

\[ \rho_{AB} \xrightarrow{\text{LOCC}} |\phi^+\rangle\langle\phi^+| \otimes m \]
Optimal Entanglement Distillation Protocol

We apply entanglement distillation to show large entropy implies large correlations.

Entanglement distillation: Given $|\psi\rangle_{ABE}$ Alice can distill $-S(A|B) = S(B) - S(AB)$ EPR pairs with Bob by making a measurement with $N \approx 2^{I(A:E)}$ elements, with $I(A:E) := S(A) + S(E) - S(AE)$, and communicating the outcome to Bob. (Devetak, Winter ‘04)
Distillation Bound

\[ S(X) > S(Y) \Rightarrow Cor(X : Z) \geq O\left(2^{-I(X:Y)}\right) \]
Distillation Bound

\[ \psi \]

\[ X \quad Y \quad Z \]

\[ l \]

\[ \text{Cor}(X : Z) \geq O \left( 2^{-I(X:Y)} \right) \]

\[ S(X) > S(Y) \Rightarrow S(X) - S(XZ) > 0 \]

(EPR pair distillation rate)

Prob. of getting one of the \( 2^{I(X:Y)} \) outcomes
Area Law from “Subvolume Law”

\[ S(X) > S(Y) \implies Cor(X : Z) \geq O\left(2^{-I(X:Y)}\right) \]
Area Law from “Subvolume Law”

\[ S(X) \leq S(Y) \iff \text{Cor}(X : Z) < O\left(2^{-I(X:Y)}\right) \]
Area Law from “Subvolume Law”

\[ S(X) \leq S(Y) \iff Cor(X : Z) < O\left(2^{-I(X:Y)}\right) \]

Suppose \( S(Y) < \frac{1}{4\xi} \) ("subvolume law" assumption)

Since \( I(X:Y) < 2S(Y) < \frac{1}{2\xi} \), a correlation length \( \xi \) implies

\[ Cor(X:Z) < 2^{-\frac{1}{\xi}} < 2^{-I(X:Y)} \]

Thus: \( S(X) < S(Y) \)
Actual Proof

We apply the bound from entanglement distillation to prove finite correlation length $\rightarrow$ Area Law in 3 steps:

- **c.** Get area law from finite correlation length under assumption there is a region with “subvolume law”
- **b.** Get region with “subvolume law” from finite corr. length and assumption there is a region of “small mutual information”
- **a.** Show there is always a region of “small mutual info”

Each step uses the assumption of finite correlation length.
Area Law in Higher Dim?

Wide open...

**Preliminary Result:** It follows from stronger notion of decay of correlations

\[ \mathbb{E}_{i_1, \ldots, i_k} \text{cor}(X : Z) \psi_{i_1, \ldots, i_k} \leq 2^{-l/\xi} \]

\( \psi_{i_1, \ldots, i_k} \) : postselected state after measurement on sites (1, ..., k) with outcomes \((i_1, \ldots, i_k)\)

Do “physical states” satisfy it??

Measurement on site \( k \)
Product States

A quantum state $\psi$ of $n$ qubits is a vector in $(\mathbb{C}^2) \otimes n \cong \mathbb{C}^{2^n}$

$$|\psi\rangle = \sum_{i_1, \ldots, i_n} c_{i_1, \ldots, i_n} |i_1, \ldots, i_n\rangle$$

Almost maximal entanglement

Exceptional Set: Low Entanglement
Product States

A quantum state $\psi$ of $n$ qubits is a vector in $(\mathbb{C}^2)^\otimes n \cong \mathbb{C}^{2^n}$

$$|\psi\rangle = \sum_{i_1, \ldots, i_n} c_{i_1, \ldots, i_n} |i_1, \ldots, i_n\rangle$$

Almost maximal entanglement

Exceptional Set: Low Entanglement

No Entanglement
Approximation Scale

We want to approximate the minimum energy (i.e. minimum eigenvalue of $H$):

$$E_0(H) = \lambda_{\text{min}} \left( \sum_{i=1}^{l} H_i \right)$$

Small total error: $E_0(H) \pm \varepsilon$

Small extensive error: $E_0(H) \pm \varepsilon l$

Are all these low-lying states entangled?
Mean-Field...

...consists in approximating the groundstate by a product state

\[ |\psi_1\rangle \otimes \ldots \otimes |\psi_n\rangle \]

\[
H(\psi_1, \ldots, \psi_n) := \sum_j \langle \psi_1, \ldots, \psi_n | H_j | \psi_1, \ldots, \psi_n \rangle
\]

It’s a mapping from quantum to classical Hamiltonians

Successful heuristic in Quantum Chemistry (Hartree-Fock)
Condensed matter

Intuition:
Mean-Field good when Many-particle interactions
Low entanglement in state
Product-State Approximation with Symmetry

- (Raggio, Werner ’89) Hamiltonians on the complete graph

\[ \tilde{H} = \mathbb{E}_{(i,j) \in G} H_{i,j} = \frac{2}{n(n-1)} \sum_{(i,j) \in G} H_{i,j} \]

\[ |E_0(\tilde{H}) - \min_\psi \langle \psi, \psi | H_{1,2} | \psi, \psi \rangle| \leq 2d^2 / n \]

- (Kraus, Lewenstein, Cirac ’12) Translational and rotational symmetric Hamiltonians in \( D \) dimensions:

\[ \tilde{H} = \mathbb{E}_{(i,j) \in G} H_{i,j} = \frac{2}{D} \sum_{(i,j) \in G} H_{i,j} \]

\[ |E_0(\tilde{H}) - \min_\psi \langle \psi, \psi | H_{1,2} | \psi, \psi \rangle| \leq 2d^2 / D \]
(B., Harrow ‘12) Let $H$ be a 2-local Hamiltonian on qudits with interaction graph $G(V, E)$ and $|E|$ local terms.

Let $\{X_i\}$ be a partition of the sites with each $X_i$ having $m$ sites.

- $E_i$: expectation over $X_i$
- $\text{Deg}$: degree of $G$
- $S(X_i)$: entropy of groundstate in $X_i$
(B., Harrow ‘12) Let $H$ be a 2-local Hamiltonian on qudits with interaction graph $G(V, E)$ and $|E|$ local terms.

Let $\{X_i\}$ be a partition of the sites with each $X_i$ having $m$ sites. Then there are states $\psi_i$ in $X_i$ s.t.

$$\frac{1}{|E|} \langle \psi_1, \ldots, \psi_n | H | \psi_1, \ldots, \psi_n \rangle \leq \frac{1}{|E|} E_0(H) + 9 \left( \frac{d^2}{\text{Deg}} \frac{\mathbb{E}_i S(X_i)}{m} \right)^{1/6}$$

$E_i$ : expectation over $X_i$

$\text{Deg}$ : degree of $G$

$S(X_i)$ : entropy of groundstate in $X_i$
Approximation in terms of degree

...shows mean field becomes exact in high dim
The Quantum PCP Conjecture

QMA-hardness theory main achievement:

(Kitaev ‘99, ...., Gottesman-Irani ‘10) Groundstates of 1D translational-invariant models are as complex as groundstates of any local Ham.

Ansatz for GS 1D TI Ham.  

Ansatz for GS any Ham.
**The Quantum PCP Conjecture**

**Quantum PCP conjecture** There are models for which all states of energy below $E_0 + \varepsilon m$ are as complex as groundstates of any local Ham.

**Ansatz with small extensive error**

(state $\psi$ for which $\langle \psi | H | \psi \rangle \leq E_0 + \varepsilon m$)

**Ansatz with small total error**

(state $\psi$ for which $\langle \psi | H | \psi \rangle \leq E_0 + 1/m_c$)

**PCP Theorem (Arora et al ‘98)** For classical Hamiltonians, to find a configuration of energy $E_0 + \varepsilon m$ is as hard as finding the minimum energy configuration.

Can we “quantize” the PCP theorem?
Approximation in terms of degree

\[
\frac{1}{|E|} \langle \psi_1, \ldots, \psi_n | H | \psi_1, \ldots, \psi_n \rangle \leq \frac{1}{|E|} E_0(H) + 9 \left( \frac{d^2}{\text{Deg}} \frac{\mathbb{E}_i S(X_i)}{m} \right)^{1/6}
\]

Implications to the quantum PCP problem:

- Limits the range of possible \( \epsilon \) for which the conjecture might be true.

- Shows that attempts to “quantize” known proofs of the classical PCP theorem (e.g. (Arad et al ’08)) cannot work.
Approximation in terms of average entanglement

\[ \frac{1}{|E|} \langle \psi_1, \ldots, \psi_n | H | \psi_1, \ldots, \psi_n \rangle \leq \frac{1}{|E|} E_0(H) + 9 \left( \frac{d^2}{\text{Deg}} \frac{\mathbb{E}_i S(X_i)}{m} \right)^{1/6} \]

Product-states do a good job if entanglement of groundstate satisfies a subvolume law:

\[ \mathbb{E}_i S(X_i) = o(m) \]

\[ m < O(\log(n)) \]
Approximation in terms of average entanglement

\[ \frac{1}{|E|} \langle \psi_1, ..., \psi_n | H | \psi_1, ..., \psi_n \rangle \leq \frac{1}{|E|} E_0(H) + 9 \left( \frac{d^2}{\text{Deg}} \frac{E_i S(X_i)}{m} \right)^{1/6} \]

If \( \mathbb{E}_i S(X_i) = \varepsilon \) we have \( O(\sqrt{\varepsilon}) \)
Approximation in terms of average entanglement

\[
\frac{1}{|E|} \langle \psi_1, ..., \psi_n | H | \psi_1, ..., \psi_n \rangle \leq \frac{1}{|E|} E_0(H) + 9 \left( \frac{d^2}{\text{Deg} \left( \mathbb{E}_i S(X_i) \right) m} \right)^{1/6}
\]

If \( \mathbb{E}_i S(X_i) = \varepsilon \) we have \( O(\sqrt{\varepsilon}) \)

In contrast, if merely \( \mathbb{E}_i S(X_i) = \varepsilon m \), the theorem shows product states give error \( O(\varepsilon^{1/6}) \)
Intuition: Monogamy of Entanglement

Quantum correlations are **non-shareable** (e.g. $(|0, 0> + |1, 1>/\sqrt{2})$)

**Idea behind QKD:** Eve cannot be correlated with Alice and Bob

Cannot be highly entangled with too many neighbors

$S(X_i)$ quantifies how much entangled $X_i$ can be with the rest

Proof uses **quantum information-theoretic techniques** to make this intuition precise
Mutual Information

1. Mutual Information \[ I(X : Y) = D(p_{XY} \parallel p_X \otimes p_Y) \]

2. Pinsker’s inequality \[ I(X : Y) = \frac{1}{2 \ln 2} \left\| p_{XY} - p_X \otimes p_Y \right\|_1^2 \]

3. Conditional MI \[ I(X : Y | Z) = I(X : YZ) - I(X : Z) \]

4. Chain Rule \[ I(X : Y_1 \ldots Y_k) = I(X : Y_1) + \ldots + I(X : Y_k | Y_1 \ldots Y_{k-1}) \]

5. Upper bound \[ I(X : Y) \leq \min(\log |X|, \log |Y|) \]

\[ 4+5 \quad \Rightarrow I(X : Y_t | Y_1 \ldots Y_{t-1}) \leq \log(|X|) / k \quad \text{for some } t \leq k \]
Quantum Mutual Information

1. Mutual Information
   \[ I(X : Y) = D(\rho_{XY} \parallel \rho_X \otimes \rho_Y) \]

2. Pinsker’s inequality
   \[ I(X : Y) = \frac{1}{2 \ln 2} \left\| \rho_{XY} - \rho_X \otimes \rho_Y \right\|_1^2 \]

3. Conditional MI
   \[ I(X : Y | Z) = I(X : YZ) - I(X : Z) \]

4. Chain Rule
   \[ I(X : Y_1 \ldots Y_k) = I(X : Y_1) + \ldots + I(X : Y_k | Y_1 \ldots Y_{k-1}) \]

5. Upper bound
   \[ I(X : Y) \leq \min(\log |X|, \log |Y|) \]

\[ 4+5 \quad \Rightarrow \quad I(X : Y_t | Y_1 \ldots Y_{t-1}) \leq \log(|X|) / k \quad \text{for some } t \leq k \]
But...

...conditioning on quantum is problematic

For $X$, $Y$, $Z$ random variables

$$I(X : Y|Z)_p = \mathbb{E}_Z I(X : Y)_{p_Z}$$

$$p_z(x, y) = p(x, y, z)/p(z)$$

No similar interpretation is known for $I(X:Y|Z)$ with quantum $Z$
Conditioning Decouples

Idea that almost works. Suppose we have a distribution $p(z_1, \ldots, z_n)$

1. Choose $i, j_1, \ldots, j_k$ at random from $\{1, \ldots, n\}$. Then there exists $t < k$ such that

$$\mathbb{E}_{i, j_1, \ldots, j_t} I(z_i : z_{j_t} \mid z_{j_1}, \ldots, z_{j_{t-1}}) \leq \frac{\log(d)}{k}$$

Define $q_{z'_{j_1}, \ldots, z'_{j_{t-1}}}(.) = p(. \mid z_{j_1} = z'_{j_1}, \ldots, z_{j_{t-1}} = z'_{j_{t-1}})$

So

$$\mathbb{E}_{j_1, \ldots, j_{t-1}} \mathbb{E}_{z'_{j_1}, \ldots, z'_{j_{t-1}}} \mathbb{E}_{i, j_t} I(z_i : z_{j_t})_q \leq \frac{\log(d)}{k}$$
Conditioning Decouples

2. Conditioning on subsystems $j_1, ..., j_t$ causes, on average, error $< k/n$ and leaves a distribution $q$ for which

$$\mathbb{E}_{i,j} I(z_i : z_j)_q \leq \frac{c \log(d)}{k} , \text{ and so } \mathbb{E}_{i \sim j} I(z_i : z_j)_q \leq \frac{n}{\text{Deg}} \frac{c \log(d)}{k}$$

By Pinsker: $\mathbb{E}_{i \sim j} \| q(z_i, z_j) - q(z_i) \otimes q(z_j) \|_1 \leq \sqrt{\frac{n}{\text{Deg}} \frac{c \log(d)}{k}}$

Choosing $k = \varepsilon n$

$$|E|^{-1} \langle H \rangle_{q_1 \otimes ... \otimes q_n} \leq \sqrt{\frac{c \log(d)}{\varepsilon \text{Deg}}} + \varepsilon$$
There exists a POVM $M(\rho) = \Sigma_k \text{tr}(M_k \rho) \ |k><k|$

s.t. for all $k$ and $\rho_{1...k}, \sigma_{1...k}$ in $D(\mathbb{C}^d \otimes k)$

$$(18d)^{-k/2} \left\| \rho_{1...k} - \sigma_{1...k} \right\|_1 \leq \left\| M^{\otimes k} (\rho_{1...k}) - M^{\otimes k} (\sigma_{1...k}) \right\|_1$$

(Lacien, Winter ‘12, Montanaro ‘12)
Proof Overview

1. Measure $\varepsilon n$ qudits with $M$ and condition on outcomes. Incur error $\varepsilon$.

2. Most pairs of other qudits would have mutual information $\leq \log(d) / \varepsilon \deg(G)$ if measured.

3. Thus their state is within distance $d^3(\log(d) / \varepsilon \deg(G))^{1/2}$ of product.

4. Witness is a global product state. Total error is $\varepsilon + d^6(\log(d) / \varepsilon \deg(G))^{1/2}$. Choose $\varepsilon$ to balance these terms.

5. General case follows by coarse graining sites (can replace $\log(d)$ by $E_i H(X_i)$)
Classical from Quantum

How the classical world we perceive emerges from quantum mechanics?

**Decoherence**: lost of coherence due to interactions with environment
Classical from Quantum

How the classical world we perceive emerges from quantum mechanics?

**Decoherence**: lost of coherence due to interactions with environment

We only learn information about a quantum system indirectly by accessing a small part of its environment. E.g. we see an object by observing a tiny fraction of its photon environment.
Quantum Darwinism in a Nutshell
(Zurek ’02; Blume-Kohout, Poulin, Riedel, Zwolak, ....)

**Objectivity of observables:** Observers accessing a quantum system by proving part of its environment can only learn about the measurement of a *preferred observer*.

**Objectivity of outcomes:** Different observes accessing different parts of the environment have almost full information about the preferred observable and *agree* on what they observe.

\[
|\phi\rangle_{B_1,\ldots,B_k} := e^{-itH_{SE}}|\psi\rangle_S \otimes |0\rangle_E
\]

\[\phi_{B_j} \text{ only contains information about the measurement of } \{M_k\}_k \text{ on } |\psi\rangle_S\]

And almost all $B_j$ have close to full information about the outcome of the measurement $\{M_k\}_k$. 
(Riedel, Zurek ‘10) Dielectric sphere interacting with photon bath: Proliferation of information about the position of the sphere

(Blume-Kohout, Zurek ‘07) Particle in brownian motion (bosonic bath): Proliferation of information about position of the particle

Is quantum Darwinism a general feature of quantum mechanics?

No: Let \( |\phi\rangle_{B_1,\ldots,B_k} := e^{-itH_{SE}} |\psi\rangle_S \otimes |0\rangle_E \)

For very mixing evolutions \( U = e^{-itH}, \phi_{B_j} \) is almost maximally mixed for \( B_j \) as big as half total system size

Information is hidden (again, QECC is an example)
Theorem (B., Piani, Horodecki ‘13) For every \( \Lambda : S \rightarrow B_1, \ldots, B_n \), there exists a measurement \( \{M_j\} \) on \( S \) such that for almost all \( k \),

\[
\Lambda_k(\rho) := \text{tr}_{B_k} \circ \Lambda(\rho) \approx \sum_j \text{tr}(M_j \rho) \sigma_{j,k}
\]

\( O(n^{-1/3}) \)

Proof by monogamy of entanglement and quantum information-theoretic techniques (similar to before)
Summary

• Thinking about entanglement from the perspective of quantum information theory is useful.

• Growing body of connections between concepts/techniques in quantum information science and other areas of physics.

Thanks!