Recitation 1

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- a) from vector spaces to Hilbert space
  + review of Dirac bra- & ket- notation
- b) tensor products & direct sums
- c) simple Harmonic oscillator
  - raising- / lowering- operators
  - number- operator
  → see e.g. Sakurai "Modern QM"
- d) measurement in QM & wave function collaps

any immediate questions?

1) Vector space

Def: A vector space over a field $\mathbb{K}$ (e.g. $\mathbb{R}, \mathbb{C}$) is a set V together with two operations that satisfy the following axioms for scalars $\alpha, \beta \in \mathbb{K}$ and vectors $\vec{u}, \vec{v}, \vec{w} \in V$

- (Addition) $+ : V \times V \to V$, s.t. $(\vec{u}, \vec{v}) \mapsto \vec{u} + \vec{v}$
  - associativity $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$
  - commutativity $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
  - identity element of addition : $\exists\ \vec{0} \in V$, called zero vector, s.t. $\forall \vec{v} \in V$
    $$\vec{v} + \vec{0} = \vec{v}$$
  - inverse element of addition : $\forall \vec{v} \in V$, $\exists$ an element $-\vec{v} \in V$, called the additive inverse of $\vec{v}$, s.t.
    $$\vec{v} + (-\vec{v}) = \vec{0}$$
- (Multiplication) $\cdot : \mathbb{K} \times V \to V$, s.t. $(\alpha, \vec{v}) \mapsto \alpha \cdot \vec{v}$
  - compatibility of scalar multiplication (associativity) : $\alpha \cdot (\beta \cdot \vec{v}) = (\alpha\beta) \cdot \vec{v}$
  - identity element of scalar multiplication : $1 \cdot \vec{v} = \vec{v}$, where $1 \in \mathbb{K}$ is the multiplicative identity
  - distributivity : $\alpha(\vec{u} + \vec{v}) = \alpha\vec{u} + \alpha\vec{v}$
    $(\alpha + \beta)\vec{u} = \alpha\vec{u} + \beta\vec{u}$

Examples: • $\mathbb{R}^n = \{ f \mid f = (x_1, \dots, x_n),\ x_1, \dots, x_n \in \mathbb{R} \}$
• $C(a,b)$: space of continuous fc[illegible]

1.1 normed space, ~~inner product space~~

- in the following we will equip our vector space with <u>additional structures</u> that allow the definition of length & angles + definition of convergence

<u>Def.</u>: A vectorspace V over a field $\mathbb{K}$ is called <u>metric space</u> if we can associate a <u>real number</u> $d(\vec{u}, \vec{v})$ (distance) (metric) to any pair of vectors $\vec{u}, \vec{v} \in V$ subject to the following axioms:

- $d(\vec{u}, \vec{v}) \geq 0$
- $d(\vec{u}, \vec{v}) = 0 \Leftrightarrow \vec{u} = \vec{v}$
- $d(\vec{u}, \vec{v}) = d(\vec{v}, \vec{u})$ "symmetry"
- $d(\vec{u}, \vec{w}) \leq d(\vec{u}, \vec{v}) + d(\vec{v}, \vec{w})$ "triangle inequality"

Tightly related is the definition of a normed space:

<u>Def.</u>: Let V be a <u>metric space</u> and simultaneously a linear space. The metric d of V is taken to be translation invariant & homogeneous. Then V is a <u>normed space</u>

$\|\vec{v}\| = d(\vec{v}, \vec{0})$ is the norm of $\vec{v}$

<u>Examples:</u> Norm in $\mathbb{R}^n$, $\mathbb{C}^n$

- In $\mathbb{R}^n$, $\mathbb{C}^n$ one can define different norms that satisfy the axioms above.
- take $p \in \mathbb{N}$:

p-norm: $\|\vec{v}\|_p := \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}$

2-norm: $\|\vec{v}\|_2 := \sqrt{\sum_{i=1}^{n} |x_i|^2}$

1-norm: $\|\vec{v}\|_1 := \sum_{i=1}^{n} |x_i|$

∞-norm: $\|\vec{v}\|_\infty := \max_{1 \le i \le n} |x_i|$

- norm in $C(a,b)$:

∞-norm: $\|\vec{v}\|_\infty := \max_{x \in [a,b]} |f(x)|$

2-norm: $\|\vec{v}\|_2 := \sqrt{\int_a^b |f(x)|^2 dx}$

1.2. unitary space / inner product space

Def: A vector space V over the field $\mathbb{C}$ is called unitary space if we can define an additional structure, called inner product or scalar product.

$V \times V \to \mathbb{C}$, s.t. for any ordered pair $u, v \in V$

we can associate a complex number $\langle u|v\rangle$

satisfying the following axioms:

For $u, v \in V$, $\alpha \in \mathbb{C}$:

- $\langle u|u\rangle \geq 0$ & real $\quad \langle u|u\rangle = 0 \Leftrightarrow u = \vec{0}$
- $\langle u|v+w\rangle = \langle u|v\rangle + \langle u|w\rangle$: linearity in the right argument
- $\langle u|\alpha v\rangle = \alpha \langle u|v\rangle$
- $\langle v|u\rangle = \langle u|v\rangle^*$

• note: ▪ Via the scalar product we can induce a norm that satisfies all the axioms of a normed space

$$||v|| := \sqrt{\langle v|v\rangle}$$

▪ Let V be unitary, for $u, v, w \in V$, $\alpha \in \mathbb{C}$, our axioms imply

$$\langle u+v|w\rangle = \langle u|w\rangle + \langle v|w\rangle \qquad \text{antilinearity}$$
$$\langle \alpha u|v\rangle = \alpha^* \langle u|v\rangle \qquad \text{in the left argument}$$

$$\langle \alpha u|v\rangle = \langle v|\alpha u\rangle^* = \alpha^* \langle v|u\rangle^* = \alpha^* \langle u|v\rangle$$

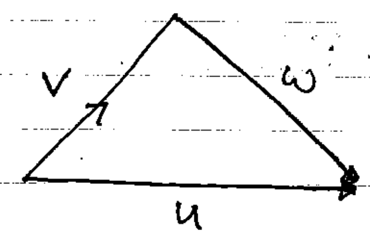
▪ for $u, v \neq \vec{0}$, we can have $\langle u|v\rangle = 0$

<u>Def</u>: 2 vectors u, v are orthogonal $u \perp v$, if

$$\langle u | v \rangle = 0$$

<u>Corollary</u>: Let V be unitary. For $u, v \in V$, we have the <u>Schwarz inequality</u>:

$$\boxed{|\langle u | v \rangle| \leq \|u\| \cdot \|v\|}$$



<u>proof</u>: choose $\omega \in V$, s.t. $u = \frac{\langle v|u\rangle}{\langle v|v\rangle} v + \omega$

$$\Rightarrow \langle v|u\rangle = \frac{\langle v|u\rangle}{\langle v|v\rangle} \langle v|v\rangle + \langle v|\omega\rangle \implies \langle v|\omega\rangle = 0$$

$$\|u\|^2 = \langle u|u\rangle = \langle u| \left(\frac{\langle v|u\rangle}{\langle v|v\rangle} |v\rangle + |\omega\rangle \right)$$

$$= \frac{\langle v|u\rangle \langle u|v\rangle}{\langle v|v\rangle} + \langle u|\omega\rangle$$

$$= \frac{\langle u|v\rangle^* \langle u|v\rangle}{\|v\|^2} + \frac{\langle v|u\rangle \overbrace{\langle v|\omega\rangle}^{=0}}{\langle v|v\rangle} + \underbrace{\langle \omega|\omega\rangle}_{=\|\omega\|^2 \geq 0}$$

$$\Rightarrow \|u\|^2 \leq \frac{\langle u|v\rangle^* \langle u|v\rangle}{\|v\|^2} \Rightarrow \boxed{\|u\|^2 \cdot \|v\|^2 \leq |\langle u|v\rangle|^2}$$

<u>corollary</u>: Let V be unitary. For $u, v \in V$, the triangle inequality holds

$$\|u + v\| \leq \|u\| + \|v\|$$

proof: $\|u+v\|^2 = \langle u+v | u+v\rangle = \langle u|u\rangle + \langle u|v\rangle + \langle v|u\rangle + \langle v|v\rangle$

$= \langle u|u\rangle + \langle v|u\rangle^* + \langle v|u\rangle + \langle v|v\rangle$

$= \langle u|u\rangle + 2\,\mathrm{Re}\langle v|u\rangle + \langle v|v\rangle \leq \langle u|u\rangle + 2|\langle v|u\rangle| + \langle v|v\rangle$

$\leq \|u\|^2 + 2\|u\| \cdot \|v\| + \|v\|^2 = (\|u\| + \|v\|)^2$

Examples: • Vectorspace $\mathbb{C}^n$. For $u = (x_1, x_2, \dots, x_n) \in \mathbb{C}^n$
$v = (y_1, y_2, \dots, y_n) \in \mathbb{C}^n$

$$\langle u | v \rangle := \sum_{i=1}^{n} x_i^* y_i$$

$\Rightarrow$ The scalar product induces the 2-norm.

• space of continuous functions on interval $[a,b]$

$f, g \in C(a,b)$

$$\langle f | g \rangle := \int_a^b f^*(x)\, g(x)\, dx$$

1.3 Hilbert space

Def: • A complete unitary space is called Hilbert space

completeness: A normed space V is complete if any Cauchy sequence has a limit in V.

dimension of vector space $V \,\hat{=}\,$ maximal number of linear independent vectors in V.
(depends on the chosen field $\mathbb{K}$).
e.g. $\mathbb{C}$ is a 1-dimensional space over $\mathbb{C}$ but a 2-dimensional space over $\mathbb{R}$

Def: Let V be a unitary space. A set $\{a_i\} \subset V$ is called orthonormal system, if for any i, j

$$\langle a_i | a_j \rangle = \delta_{ij}$$

Def: Let $\mathcal{H}$ be a Hilbert-space. A orthonormal system $\{a_i\} \subset \mathcal{H}$ is complete if $\nexists\, v \in \mathcal{H} \setminus \{\vec{0}\}$, s.t. $\langle v | a_i \rangle = 0 \quad \forall\, |a_i\rangle$

1:4 Linear Functionals & dual space

<u>Def</u>: Let $\mathcal{H}$ be a Hilbert space, $\mathbb{K}$ a field and $D \subset \mathcal{H}$.
A <u>linear map</u> $L : D \to \mathbb{K}$, $v \mapsto L(v)$ is called <u>linear functional</u>.

The set $D(L) = D$ is the <u>domain</u> of L, the set $W(L) = L(D(L))$ is the <u>image</u> of L.

For any $u, v \in D(L)$, $\alpha, \beta \in \mathbb{K}$, we have:

$$L(\alpha u + \beta v) = \alpha L(u) + \beta L(v)$$

The norm of L is defined as:

$$\|L\| := \sup_{\substack{f \in D(L) \\ \|f\| = 1}} \|L(f)\|$$

<u>note</u>: functional is continuous, if bounded.

<u>important theorem (Riesz)</u>:

Let $\mathcal{H}$ be a Hilbert space. Associated to any linear, continuous functional L with $D(L) = \mathcal{H}$ is exactly one element $g_L \in \mathcal{H}$, s.t. the functional can be written as scalar product.

$$L(f) = \langle g_L | f \rangle \qquad \forall f \in \mathcal{H}$$

$$\|L\| = \|g_L\|$$

The set of linear, bounded functionals on a Hilbert space has Hilbert space-structure.

Def: Let $\mathcal{H}$ be a Hilbert space and $\mathbb{K}$ be a field.
The set $\tilde{\mathcal{H}}$ of linear, bounded functionals $\mathcal{H} \to \mathbb{K}$ is called the dual space of $\mathcal{H}$.

Denote $\tilde{g} = L_g$ the linear functional s.t.

$$f \xmapsto{\tilde{g}} \langle g | f \rangle$$

In $\tilde{\mathcal{H}}$, we have the following operations

$\beta \in \mathbb{C}$
$\tilde{f}, \tilde{g}, \tilde{g}_1, \tilde{g}_2 \in \tilde{\mathcal{H}}$
A: operator

$$f \xmapsto{\tilde{g}_1 + \tilde{g}_2} \langle g_1 + g_2 | f \rangle$$

$$f \xmapsto{\widetilde{\beta g}} \langle \beta g | f \rangle = \beta^* \langle g | f \rangle$$

$$f \xmapsto{\widetilde{Ag}} \langle Ag | f \rangle = \langle g | A^+ f \rangle$$

and the scalar product

$$\langle \underset{\tilde{\mathcal{H}}}{\underset{\cup\!\!\!\!\!\!\!\mid}{\tilde{f}}} | \underset{\tilde{\mathcal{H}}}{\underset{\cup\!\!\!\!\!\!\!\mid}{\tilde{g}}} \rangle = \langle \underset{\mathcal{H}}{\underset{\cup\!\!\!\!\!\!\!\mid}{g}} | \underset{\mathcal{H}}{\underset{\cup\!\!\!\!\!\!\!\mid}{f}} \rangle = \langle f | g \rangle^*$$

$\mathcal{H}$ and $\tilde{\mathcal{H}}$ are isomorphic

note: $\tilde{g}$ is the dual vector of g.

Dirac - Notation

- Riesz-theorem: linear functionals can be represented as scalar products.

- notation: $|f\rangle, |g\rangle \in \mathcal{H}$ "ket-vectors"

  $\langle f|, \langle g| \in \tilde{\mathcal{H}}$ "bra-vectors"
  - (instead of $\tilde{f}, \tilde{g}$)
  - linear functionals

$$L_g(f) = \langle g|f\rangle = \langle f|g\rangle^*$$

$$L_g(Af) = \langle g|Af\rangle =: \langle g|A|f\rangle$$

$$L_{\beta f}(g) = \langle \beta f|g\rangle = \beta^* \langle f|g\rangle$$

$$L_{Af}(g) = \langle Af|g\rangle = \langle f|A^+g\rangle = \langle f|A^+|g\rangle$$

- $\longleftrightarrow$: "is dual to":

$$\langle f| \longleftrightarrow |f\rangle$$
$$\beta^* \langle f| \longleftrightarrow \beta|f\rangle$$
$$\langle f|A^+ \longleftrightarrow A|f\rangle$$

c.f. Hitoshi Murayama lecture notes (Berkeley) Ph 221A

b) Direct sum & tensor products

Direct sum:

Way of getting a new big vector space from 2 (or more) smaller vector spaces.

start with : V : n-dim. vector space

W : m-dim. vector space

$\{\vec{e}_1, \dots, \vec{e}_n\}$: basis of V
$\{\vec{f}_1, \dots, \vec{f}_m\}$: basis of W.
$\Rightarrow$ define $n+m$ basis vectors

$$\underbrace{\{\vec{e}_1, \dots, \vec{e}_n, \vec{f}_1, \dots, \vec{f}_m\}}$$

they span: $\underline{V \oplus W}$

"direct sum of V & W"

- any vector $\vec{v} \in V$ has decomposition

$$\vec{v} = \sum_{i=1}^{n} v^i \vec{e}_i \qquad \vec{v} = \left.\begin{pmatrix} v^1 \\ \vdots \\ v^n \end{pmatrix}_V \right\} n$$

element of V

$$\vec{w} = \sum_{j=1}^{m} w^j \vec{f}_j \qquad \vec{w} = \left.\begin{pmatrix} w^1 \\ \vdots \\ w^m \end{pmatrix}_W \right\} m$$

element of W

$\Rightarrow$ vectors are naturally elements of the direct sum by filling in zeros!

$$\vec{V} = \left.\begin{pmatrix} v^1 \\ \vdots \\ v^n \\ 0 \\ \vdots \\ 0 \end{pmatrix}\right\} n+m \qquad \vec{w} = \left.\begin{pmatrix} 0 \\ \vdots \\ 0 \\ w^1 \\ \vdots \\ w^m \end{pmatrix}\right\} n+m \Rightarrow \vec{V} \oplus \vec{w} = \begin{pmatrix} v^1 \\ \vdots \\ v^n \\ w^1 \\ \vdots \end{pmatrix}$$

- focus on linear maps $A : V \to V$;
$$\vec{v} \mapsto A\vec{v}$$

- look @ matrix representation of A

$$\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}_V \mapsto \underbrace{\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}}_{\equiv A} \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}_V$$

- similarly there is a map $B : W \to W$
$$\vec{w} \mapsto B \cdot \vec{w}$$

- On the direct sum space, the same matrices can still act on the respective vectors

- this map is written as $A \oplus B : V \oplus W \mapsto V \oplus W$

by the following form

$$A \oplus B = \left(\begin{array}{c|c} A_{n \times n} & 0_{n \times m} \\ \hline 0_{m \times n} & B_{m \times m} \end{array} \right)$$

$$(A \oplus B) \cdot (\vec{v} \oplus \vec{w}) = \left(\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array} \right) \left(\frac{\vec{v}}{\vec{w}} \right) = \left(\frac{A\vec{v}}{B\vec{w}} \right)$$
$$= (A\vec{v}) \oplus (B\vec{w})$$

- if we have 2 matrices, their multiplication is done as follows
$$(A_1 \oplus B_1)(A_2 \oplus B_2) = (A_1 A_2) \oplus (B_1 B_2)$$

- note that not every matrix on $V \oplus W$ can be written as a direct sum of a matrix on V and another on W.

- simple counting:
  - there are $(n+m)^2$ independent matrices on $V \oplus W$, while there are only n^2 on V and m^2 on W.
  - the remaining $(n+m)^2 - n^2 - m^2 = 2nm$ matrices can <u>not</u> be written as a direct sum.

- useful formulae:

$$\det(A \oplus B) = (\det A)(\det B)$$
$$\mathrm{tr}[A \oplus B] = (\mathrm{tr}[A]) + (\mathrm{tr}[B])$$

Tensor Product (direct product, Kronecker product)

- another way of constructing a bigger vector space out of smaller vector spaces.

- start with 2 vector spaces $\underset{n\text{-dim}}{V}$ & $\underset{m\text{-dim}}{W}$

$$V \otimes W \rightarrow (n \cdot m)\text{-dimensional vector space.}$$

- again $\{\vec{e}_1, \dots \vec{e}_n\}$ basis of V
  $\{\vec{f}_1, \dots \vec{f}_m\}$ basis of W

vectors:

- now define $n \cdot m$ basis vectors $\vec{e}_i \otimes \vec{f}_j$ where $i = 1, \dots, n$, $j = 1, \dots, m$

  - we will be more explicit in a moment.

  - regard these objects as new set of basis vectors.
  - "tensor" $\leftrightarrow$ basis elements have 2 indices (i,j)

- explicit example: $n = 2$, $m = 3$ $\Rightarrow$ tensor product space $n \cdot m = 6$-dim.

$$\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \vec{f}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \vec{f}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad \vec{f}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\vec{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$\Rightarrow$ new basis vectors: $\vec{e}_1 \otimes \vec{f}_1 = \begin{pmatrix} 1 \times \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\ \cdots \\ 0 \times \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \cdots \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \dots \quad \vec{e}_2 \otimes \vec{f}_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \cdots \\ 0 \\ 0 \\ 1 \end{pmatrix}$

$\vec{e}_1 \otimes \vec{f}_2 = \begin{pmatrix} 1 \times \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \\ \cdots \\ 0 \times \begin{pmatrix} 0 \\ 1 \\ \cdots \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \cdots \\ 0 \end{pmatrix}$

- generally: $\vec{v} \otimes \vec{w} = \begin{pmatrix} v_1 w_1 \\ v_1 w_2 \\ v_1 w_3 \\ \hdashline v_2 w_1 \\ v_2 w_2 \\ v_2 w_3 \end{pmatrix}$

- <u>maps / operators / matrices</u>:

map A: $\vec{v} \mapsto A\vec{v}$ $\qquad \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}_V \longmapsto \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}_V$

- on the tensor product space, the same matrix can still act on the vectors

$$\vec{v} \mapsto A\vec{v} \quad , \quad \vec{w} \mapsto \vec{w} \text{ untouched}$$

- this map is written as $A \otimes I$ ← identity operator

example $n = 2$ $\quad m = 3$

A is (2×2) $\qquad A \otimes I$ is (6×6)

$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \qquad I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$A \otimes I = \begin{pmatrix} a_{11}\begin{pmatrix} 1&0&0\\0&1&0\\0&0&1 \end{pmatrix} & a_{12}\begin{pmatrix} 1&0&0\\0&1&0\\0&0&1 \end{pmatrix} \\ a_{21}\begin{pmatrix} 1&0&0\\0&1&0\\0&0&1 \end{pmatrix} & a_{22}\begin{pmatrix} 1&0&0\\0&1&0\\0&0&1 \end{pmatrix} \end{pmatrix} = \left(\begin{array}{ccc|ccc} a_{11} & 0 & 0 & a_{12} & 0 & 0 \\ 0 & a_{11} & 0 & 0 & a_{12} & 0 \\ 0 & 0 & a_{11} & 0 & 0 & a_{12} \\ \hline a_{21} & 0 & 0 & a_{22} & 0 & 0 \\ 0 & a_{21} & 0 & 0 & a_{22} & 0 \\ 0 & 0 & a_{21} & 0 & 0 & a_{22} \end{array}\right)$$

$\Rightarrow$ can act on $\vec{v} \otimes \vec{w}$: $(A \otimes I)(\vec{v} \otimes \vec{w}) = (A\vec{v}) \otimes (\vec{w})$

- similar with $\mathbb{1} \otimes B$!

• in general:

$$(A\otimes I)(\vec{v}\otimes\vec{w}) = (A\vec{v})\otimes\vec{w}$$
$$(I\otimes B)(\vec{v}\otimes\vec{w}) = \vec{v}\times(B\cdot\vec{w})$$

$$(A_1\otimes I)(A_2\otimes I) = (A_1\cdot A_2)\otimes I$$

$$(A\otimes I)(I\otimes B) = (I\otimes B)(A\otimes I) = (A\otimes B)$$

$$(A\otimes B)(\vec{v}\otimes\vec{w}) = (A\vec{v})\otimes(B\vec{w})$$

example: $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ $\qquad B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$

$$A\otimes B = \left(\begin{array}{c|c} a_{11}\cdot\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ \vdots & & \vdots \\ b_{31} & \cdots & b_{33}\end{pmatrix} & a_{12}\cdot B \\ \hline a_{21}\cdot B & a_{22}\, B \end{array}\right) \nwarrow 6\times6$$

useful formulae: $\det(A\otimes B) = (\det A)^m (\det B)^n$

$$\mathrm{tr}[A\otimes B] = (\mathrm{tr}A)\cdot(\mathrm{tr}\,B)$$

tensor product in QM

- QM: associate Hilbert space for each dynamical d.o.f.

o e.g. free particle in 3d $\{P_x, P_y, P_z\}$

- note can only specify p_x or x not both $\Rightarrow$ 3 d.o.f.
- momentum eigenstates

$$P_x |p_x\rangle = p_x |p_x\rangle \quad \text{form a basis of } \mathcal{H}_x$$

- full eigenstate of complete Hamiltonian is a tensor product

$$|p_x, p_y, p_z\rangle = |p_x\rangle \otimes |p_y\rangle \otimes |p_z\rangle$$

- usually we don't bother writing it this way, but P_x is then $P_x \otimes \mathbb{1} \otimes \mathbb{1}$ and acts as

$$(P_x \otimes \mathbb{1} \otimes \mathbb{1})\, |p_x\rangle \otimes |p_y\rangle \otimes |p_z\rangle = (P_x |p_x\rangle) \otimes |p_y\rangle \otimes |p_z\rangle$$

o important for spin & angular momentum

Measurement in QM

axioms of QM:

I) physical state is represented by state $|\psi\rangle$ in Hilbert space

II) Observables are represented by Hermitian operators. Functions of obs. are represented by fcts. of herm. ops.

III) The expectation value of an observable A is given by
$$\langle\psi|A|\psi\rangle$$

IV) dynamic is given by Schrödinger equation
$$i\hbar\,\partial_t|\psi\rangle = \hat{H}|\psi\rangle$$

V) after a measurement of observable A with result a, the initial state "collapses" into $|a\rangle$

Formalization of measurements

- QM measurements described by set of measurement operators
$$\{M_m\}$$
- state before measurement $|\psi\rangle$, result m occurs with probability
$$p(m) = \langle\psi|M_m^{+}M_m|\psi\rangle \qquad p(m)\geq 0$$
- state after measurement $|\psi_m\rangle = \dfrac{M_m|\psi\rangle}{\sqrt{\langle\psi|M_m^{+}M_m|\psi\rangle}}$

• completeness of M_m-operators

$$\sum_m M_m^\dagger M_m = \mathbb{1} \iff 1 = \sum_m p(m) = \sum_m \langle\psi|M_m^\dagger M_m|\psi\rangle$$

example: Q-bit system $\left\{ \begin{matrix} |0\rangle \\ |1\rangle \end{matrix} \right\}$

measurement operators $M_0 = |0\rangle\langle 0|$ $\qquad M_1 = |1\rangle\langle 1|$

M_α : hermitian
$M_\alpha^2 = M_\alpha$

(completeness) : $M_0^\dagger M_0 + M_1^\dagger M_1 = M_0 + M_1 = \mathbb{1}$

• start with state $|\psi\rangle = a|0\rangle + b|1\rangle$

$$p(0) = \langle\psi| M_0^\dagger M_0 |\psi\rangle = \langle\psi|M_0|\psi\rangle = |a|^2$$

$$p(1) = |b|^2$$

$$\Rightarrow |\psi_0\rangle = \frac{M_0|\psi\rangle}{|a|} = \frac{a}{|a|}|0\rangle$$

pure phase

$$|\psi_1\rangle = \frac{M_1|\psi\rangle}{|b|} = \frac{b}{|b|}|1\rangle$$

projective measurement (von Neumann - measurement)

(special case of measurement postulate)

- A P.M is described by observable M (hermitian)

$$M = \sum_m m \, P_m$$

projector on eigenstate $|m\rangle$ associated to eigenvalue m

- measuring in state $|\psi\rangle$, the probability to get m, is given by:

$$p(m) = \langle \psi | P_m | \psi \rangle$$

state after measurement $\quad |\psi_m\rangle = \dfrac{P_m |\psi\rangle}{\sqrt{p(m)}}$

Projective measurement follows from general case if $\{M_m\}$ are orthogonal projectors, i.e. M_m hermitian & $M_m M_{m'} = \delta_{mm'} M_m$