Ph125: Problem Set 1 Solutions

October 13, 2016

Problem 1

Note that,

$$|3\rangle = \begin{pmatrix} 2 & 2 \\ 0 & -2 \end{pmatrix} = 2\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 2 & 2 \\ -2 & -2 \end{pmatrix} = 2|1\rangle + |2\rangle \ .$$

Therefore, vector $|3\rangle$ is a linear combination of vectors $|1\rangle$ and $|2\rangle$. This indicates that $|1\rangle, |2\rangle$ and $|3\rangle$ are not linearly independent.

Problem 2

(a) The characteristic equation is,

$$\det[\Lambda - \lambda I] = \begin{vmatrix} 4-\lambda & 1 \\ -1 & 2-\lambda \end{vmatrix} = 0 \ ,$$

which gives,

$$(\lambda - 3)^2 = 0 \ ,$$

with roots $\lambda_1 = \lambda_2 = 3$. Those are the eigenvalues of the matrix Λ. The eigenspace is two-fold degenerate. Since $\lambda = 3$ is the only eigenvalue, we find that the components x_1 and x_2 of the corresponding eigenvectors must obey the matrix equation

$$\begin{pmatrix} 4-3 & 1 \\ -1 & 2-3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \mathbf{0} \ .$$

It is now evident that all eigenvectors are of the form $\begin{pmatrix} a \\ -a \end{pmatrix}$.

(b) This time the characteristic equation has the form,

$$\det[\Omega - \lambda I] = \frac{1}{2} \begin{vmatrix} 2-2\lambda & 0 & 0 \\ 0 & 3-2\lambda & -1 \\ 0 & -1 & 3-2\lambda \end{vmatrix} = 0 \ ,$$

i.e.,

$$(\lambda - 1)^2 (\lambda - 2) = 0 \ .$$

Thus the eigenvalues are $\lambda_1 = \lambda_2 = 1$, $\lambda_3 = 2$. The space spanned by eigenstates corresponding to eigenvalues λ_1 and λ_2 is degenerate. For $\lambda = 1$ we get,

$$\frac{1}{2}\begin{pmatrix} 2-2\cdot 1 & 0 & 0 \\ 0 & 3-2\cdot 1 & -1 \\ 0 & -1 & 3-2\cdot 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \mathbf{0} \ ,$$

which yields the first two normalized eigenvectors,

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{v}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \ .$$

For $\lambda = 2$ the third normalized eigenvector turns out to be,

$$\vec{v}_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

Problem 3

(a) Take $i = j$, we have $M^{i2} = I$. Since the eigenvectors of a Hermitian matrix form a complete basis, we can write M^i as $U^{i\dagger}D^iU^i$, where D^i is diagonal and U^i is unitary. The condition $M^{i2} = I$ gives

$$\begin{aligned} M^{i2} &= U^{i\dagger}D^{i2}U^i = I, \\ D^{i2} &= \mathrm{Diag}(\lambda_1^2, \lambda_2^2, ..., \lambda_i^2, ...) = I, \\ \lambda_i^2 &= 1, \forall i. \\ \lambda_i &= \pm 1, \forall i. \end{aligned}$$

(b) For a given matrix M^i, pick another matrix M^j ($j \neq i$). Use $M^iM^j = -M^jM^i$, $M^{j2} = I$ and cyclic property of trace, we have

$$\begin{aligned} \mathrm{Tr}(M^i) &= \mathrm{Tr}(M^iM^jM^j) \\ &= \mathrm{Tr}(M^jM^iM^j) \\ &= \mathrm{Tr}(-M^iM^jM^j) = -\mathrm{Tr}(M^i), \end{aligned}$$

where we use the cyclic property of trace in the second equality and $M^iM^j = -M^jM^i$ in the thrid equality. It is clear that $\mathrm{Tr}(M^i)$ has to be zero, i.e., M^i is traceless.

(c) The trace is related to eigenvalues by

$$\begin{aligned} \mathrm{Tr}(M^i) &= \mathrm{Tr}(U^{i\dagger}D^iU^i) = \mathrm{Tr}(U^iU^{i\dagger}D^i) \\ &= \mathrm{Tr}(D^i) = \sum_i \lambda_i. \end{aligned}$$

The vanishing trace implies the sum of eigenvalues are zero. However, we know from a.) that $\lambda_i = \pm 1$. The only case for $\sum_i \lambda_i = 0$ is that the number of $+1$ eigenvalues equals to the number of -1. Therefore, the dimension of the matrices, which is the total number of eigenvalues, must be even.

Problem 4

The function g_Δ is exactly the Gaussian distribution. From the property of Gaussian distribution, it satisfies

$$\int_{-\infty}^{+\infty} g_\Delta(x-x')dx' = 1.$$

As $\Delta \to 0$, the width of the Gaussian distribution goes to zero. Thus for $x \neq x'$, $g_\Delta(x-x')$ approaches to zero. (One can show it rigorously by taking limit of $e^{-1/\Delta^2}/\Delta$.) Since $g_\Delta(x-x')$ is nonzero only at $x = x'$, the above integral in the limit of $\Delta \to 0$ can be generalize to

$$\int_{-\infty}^{+\infty} g_\Delta(x-x')dx' = \int_b^a g_\Delta(x-x')dx' = 1,$$

where $a > x > b$. We find g_Δ in the limit of $\Delta \to 0$ satisfies the properties of delta function.

One can prove $\int_b^a f(x')g_\Delta(x-x')dx' = f(x)$ by Taylor expanding the test function $f(x')$ around x. All the terms after integration over x' scale with Δ to positive powers, except for the constant term $f(x)$. Thus, the integration goes to the result of Dirac delta function in the limit of $\Delta \to 0$. (This part is not required for full credit.)

Problem 5

(a) We define the addition and multiplication as the usual operation on functions. The differentiation is also a linear operator. Obiviously, all the axioms of vector space are satified.

However, we need to check if the boundary condition $f(0) = f(2\pi) = 0$ still holds under addition and multiplication.

For $f, g, h \in V$ and $a, b \in \mathcal{R}$, we have

- Commutativity: $f(x)+g(x) = g(x)+f(x) \equiv [f+g](x)$. Note that $[f+g](x)$ is still real, twice differentiable, and $[f+g](0) = f(0)+g(0) = 0$ (similarly for $[f+g](2\pi)$). Thus, the addition is close.
- Associativity: $(f(x)+g(x))+h(x) = f(x)+(g(x)+h(x))$
- Identity: $I(x) \equiv 0$ is the identity of the vector space. $f(x)+I(x) = I(x)+f(x) = f(x)$, $\forall f \in V$.
- Inverse: For every $f(x)$, define $f^{-1}(x) \equiv -f(x)$. We find $f(x)+f^{-1}(x) = f^{-1}(x)+f(x) = 0 = I(x)$.
- Associativity of multiplication: $a(bf(x)) = b(af(x))$
- Distributivity of scalar sums: $(a+b)f(x) = af(x)+bf(x)$
- Distributivity of vector sums: $a(f(x)+g(x)) = af(x)+ag(x)$
- Distributivity of vector sums: $1 \cdot f(x) = f(x)$

Note that the multiplication is also close. $af(x)$ is real, twice differentiable, and vanishes at 0 and 2π.

(b) The axioms of an inner product are

- $\langle f|g\rangle = \langle g|f\rangle^*$: By definition we know $\langle f|g\rangle = \int_0^{2\pi} f(x)g(x)dx = \int_0^{2\pi}(g(x)f(x))^*dx = \langle g|f\rangle^*$. The functions are real so their complex conjugate are the same.

– Linearity: For $a, b \in \mathcal{R}$ and $f, g, h \in V$, $\langle af + bg|h\rangle = \int_0^{2\pi}(af(x) + bg(x))h(x) = \int_0^{2\pi}(af(x)h(x) + bg(x)h(x)dx = a\langle f|h\rangle + b\langle g|h\rangle$

– Positive definiteness: $\langle f|f\rangle = \int_0^{2\pi} f^2(x)dx \geq 0$, since f is real. And $\langle f|f\rangle = 0$ only happens for $f(x) = 0 = I(x)$.

(c) To check whether an operator is Hermitian or not in infinite dimensional vector space, we need to verify that the matrix element satisfies

$$\langle f|\hat{\mathcal{O}}|g\rangle = \langle f|\hat{\mathcal{O}}^\dagger|g\rangle$$
$$\langle f|\hat{\mathcal{O}}g\rangle = \langle g|\hat{\mathcal{O}}|f\rangle^*$$
$$\langle f|\hat{\mathcal{O}}g\rangle = \langle g|\hat{\mathcal{O}}f\rangle^*.$$

In our definition of inner product, this means we need to check

$$\int_0^{2\pi} f(x)\frac{d^2g}{dx^2}dx = (\int_0^{2\pi} g(x)\frac{d^2f}{dx^2}dx)^*.$$

Proof:

$$\begin{aligned}\int_0^{2\pi} f\frac{d^2g}{dx^2}dx &= f\frac{dg}{dx}|_0^{2\pi} - \int_0^{2\pi}\frac{df}{dx}\frac{dg}{dx}dx \\ &= -\int_0^{2\pi}\frac{df}{dx}\frac{dg}{dx}dx \\ &= \int_0^{2\pi} g\frac{d^2f}{dx^2}dx \\ &= (\int_0^{2\pi} g\frac{d^2f}{dx^2}dx)^*.\end{aligned}$$

The boundary term vanishes by boundary condition. In the third equality, we use the fact that the second line is symmetric of f and g, so it equals the term in right-hand-side but with f and g swapped. The final equality is again from the reality of the functions.

(d) The eigenvalue problem is the same as solving the differential equation

$$\hat{\mathcal{O}}f(x) = \frac{d^2f}{dx^2} = \lambda f(x),$$

with boudary condition $f(0) = f(2\pi) = 0$.

The nontrivial solution is $F_n(x) = \sin(\omega_n x)$ with $\lambda = -\omega_n^2$, where $\omega_n = n/2, n \in \mathcal{N}$ is fixed by the condition $f(2\pi) = 0$. Note that $\omega = 0$ gives trivial eigenfunction and $\omega = -1/2, -1, -3/2, ...$ corresponds to the same eigenfunctions as ω_n, so they shouldn't be included.

Note that hyperbolic functions solve the differential equation but doesn't satisfy the boundary condition.

(e) For $n \neq m$,

$$\begin{aligned}\langle F_n(x)|F_m(x)\rangle &= \int_0^{2\pi} \sin(n/2x)\sin(m/2x)dx \\ &= 2\int_0^{\pi} \sin(ny)\sin(my)dy \\ &= \int_0^{\pi} \cos((n-m)y) - cos((n+m)y)dy \\ &= [\frac{1}{n-m}\sin((n-m)y) - \frac{1}{n+m}\sin((n+m)y)]|_0^{\pi}\end{aligned}$$

This is zero if $n \neq m$.

For $n = m$, we can use the fact that $\sin^2(x)$ has average 1/2. Therefore,

$$\begin{aligned}\langle F_n(x)|F_n(x)\rangle &= \int_0^{2\pi} \sin^2(n/2x)dx \\ &= \frac{1}{2}2\pi = \pi.\end{aligned}$$

To normalize F_n, we rescale it to be

$$F_n(x) = \frac{1}{\sqrt{\pi}}\sin(\frac{n}{2}x), n \in \mathcal{N}.$$

(f) We can extend the definition of $f(x)$ to $[-2\pi, 0]$ by defining $f(-x) \equiv -f(x)$ for $x \in [0, 2\pi]$. The Fourier theorem states that any periodic function in $[-2\pi, 2\pi]$ can be written as

$$f(x) = \frac{a_0}{2} + \sum a_n \sin(n/2x) + \sum b_n \cos(n/2x).$$

Given the boundary condition $f(0) = f(\pm 2\pi) = 0$, we find $a_0 = b_n = 0$. So every function in V can be written as a series of sine functions, which are the eigenfunctions $F_n(x)$.

(g) We know $f(x) = \sum_n a_n F_n(x)$ and $\langle F_n(x)|F_m(x)\rangle = \delta_{nm}$. The norm of $f(x)$ reads

$$\begin{aligned}|f(x)|^2 &= \langle f(x)|f(x)\rangle \\ &= \sum_{n,m} a_n a_m \langle F_n(x)|F_m(x)\rangle \\ &= \sum_{n,m} a_n a_m \delta_{nm} \\ &= \sum_n a_n^2.\end{aligned}$$

This gives $|f(x)| = \sqrt{\sum_n a_n^2}$.